7.10.14 a)

Let $A = \begin{matrix} a & b \\ c & d \end{matrix}$

A is orthogonal

$\therefore a^2+b^2=1$

$a^2+c^2=1$

$c^2+d^2=1$

$b^2+d^2=1$

$\rightarrow a^2=d^2,\ b^2=c^2$

$d=\pm a \quad c=\pm b$

$\det(A)=ad-bc=1=a^2+b^2$

$a(\pm a)-b(\pm b)=a^2+b^2$

$\therefore d=+a,\ c=-b$

$A=\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$

and $\theta = \arcsin(a) = \arccos(b)$

b) Follow program above, but

$\det(A)=ad-bc=-1$

$a(\pm a)-b(\pm b)=-1$

Since $a^2+b^2=1$, $-a^2-b^2=-1$,

$\therefore d=-a,\ c=+b$

$A=\begin{bmatrix} a & b \\ b & -a \end{bmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ +\sin\theta & -\cos\theta \end{pmatrix}$

$\theta = \arccos(a) = \arcsin(b)$

Since the norms of the rows and columns of A in both cases are 1, the entries of A always satisfy formulation in these terms of sin, cos

7.10.6

a) take $A=1$, $B=1$

$AA^*=1$, $B \cdot B^*=1$

$A+B=2$, $(A+B)^*=2$

$(A+B)(A+B)^*=4\neq 1$

$\therefore$ A+B not necessarily unitary

b) $(A+B)(A+B)^*$

$= AA^*+AB^*+BA^*+BB^*$

$= 2I+AB^*+BA^*$

If $(A+B)$ not unitary

$2I+AB^*+BA^*\neq I$

$AB^*+BA^*\neq -I$

and $AB^*AB^*+I\neq AB^*$

Since the product of unitary matrices is unitary, AB^*AB^* is unitary. We know AB^* is unitary. The addition of the identity to any unitary matrix results in a non-unitary matrix. (Adding 1 to an entry of a $v: \|v\|=1$

6.10.6 cont'd

c) $(AB)(AB)^* = ABB^*A^* = AIA^* = AA^* = I$

$\therefore$ AB unitary

d) $(AB)(AB)^* = I$, $(AB)^*(AB) = I$

~~ABABABAB=I~~

$B^*A^*AB = I$

$B^*IB = I$

$B^*B = I$

$BB^* = I$

$\therefore$ B is unitary

7.4.7

$3x_1^2 + 4x_1x_2 + 8x_1x_3 + 3x_3^2$

$= \sum_{i=1}^{m=3}\sum_{j=1}^{n=3} a_{ij}x_ix_j = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{12}+a_{21})x_1x_2$
$+ (a_{13}+a_{31})x_1x_3 + (a_{23}+a_{32})x_2x_3$

$a_{11} = 3$ $\quad a_{12}, a_{21} = 2$
$a_{22} = 0$ $\quad a_{13}, a_{31} = 4$
$a_{33} = 3$ $\quad a_{23}, a_{32} = 2$

$\Rightarrow A = \begin{matrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{matrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$

$= (\lambda+1)^2(\lambda-8) = 0$

$\lambda = -1, 8$

$\lambda = 8$

$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow$ $\begin{matrix} 3x_1 + 2x_2 + 4x_3 = 8x_1 \\ 2x_1 + 2x_3 = 8x_2 \\ 4x_1 + 2x_2 + 3x_3 = 8x_3 \end{matrix}$

$x_1 = x_3$, $x_1 = 2x_2$ $\quad \underset{\rightarrow}{e}_8 = (1/3)\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$\lambda = -1$

$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\begin{matrix} 3x_1 + 2x_2 + 4x_3 = -x_1 \\ 2x_1 + 2x_3 = -x_2 \\ 4x_1 + 2x_2 + 3x_3 = -x_3 \end{matrix}$ $\Rightarrow$ $x_1 = (-1/2)x_2 - x_3$
$x_2 = x_2$, $x_3 = x_3$

$\underset{\rightarrow}{e}_{-1} = x_2\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Adjusting the two components of $\underset{\rightarrow}{e}_{-1}$ and $\underset{\rightarrow}{e}_8$ so that their norm is $\sqrt{90}$, we can form C

$\sqrt{90} = 3\sqrt{10}$

$C = \frac{1}{3\sqrt{10}}\begin{bmatrix} 2\sqrt{10} & -3\sqrt{2} & -3\sqrt{5} \\ \sqrt{10} & 6\sqrt{2} & 0 \\ 2\sqrt{10} & 0 & 3\sqrt{5} \end{bmatrix}$

7.14.17

$x^2 + 4xy + 2y^2 - 12 = 0$

$x_1^2 + 4x_1x_2 + 2x_2^2 - 12 = 0$

$\sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}x_1x_2 = a_{11}x_1^2 + (a_{12}+a_{21})x_1x_2 + a_{22}x_2^2 = x_1^2 + 4x_1x_2 + 2x_2^2$

$a_{11}=1, \quad a_{22}=-2, \quad a_{12}, a_{21} = 2$

$A = \begin{matrix} 1 & 2 \\ 2 & -2 \end{matrix}$ $\quad \det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda+2 \end{vmatrix} = (\lambda-2)(\lambda+3)$

$\lambda = -3, 2$

$\Rightarrow y^T \Delta y = \sum_{i=1}^{2} \lambda_i y_i^2 = 2y_1^2 - 3y_2^2$

$\Rightarrow 2y_1^2 - 3y_2^2 - 12 = 0$

$\lambda_1\lambda_2 < 0$ Hyperbola at $(0,0)$