Math 2046
HW2: (1.15: 15, 1.15: 20, 1.17: 3, 3.5: 2, 3.5: 10)
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1.15: 15. Given three linearly independent vectors in *A*, *B*, *C* in $\mathbf{R}^n$. Prove or disprove each of the following:

(a) $A+B, B+C, A+C$ are linearly independent.

Proof (by contradiction). Assume linear dependence. Then we can write:

$$x(A+B)+y(B+C)+z(A+C)=0$$
$$x, y, z \in \mathbb{R}, x, y, z \neq 0$$

or, another way,

$$A(x+z)+B(x+y)+C(y+z)=0$$
$$(x+z, x+y, y+z) \neq (0,0,0).$$

Because *A*, *B*, *C* are linearly independent, no nontrivial linear combination of *A*, *B*, *C* can equal 0. Thus, the above equation has no solutions.
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(b) $A-B, B+C, A+C$ are linearly independent.

Counterexample. $(A-B)+(B+C)-(A+C)=0$.

1.15: 20. Let *A* and *B* denote two finite subsets of vectors in $\mathbf{R}^n$, and let *L*(*A*) and *L*(*B*) denote their linear spans. Prove each of the following statements:

(a) If $A \subseteq B$, then $L(A) \subseteq L(B)$.

Proof. Let $A=\{A_0, \ldots, A_n\}$ and $B=\{B_0, \ldots, B_n, B_{(n+1)}, \ldots, B_p\}$, with $(A_0 \ldots A_n)=(B_0 \ldots B_n)$.

$$x \in L(A) \Rightarrow \exists c_i \in \mathbb{R},\ x=\sum_{i=0}^{n} c_i A_i, \text{ so } x=\sum_{i=0}^{n} c_i A_i + \sum_{i=n+1}^{p} 0 B_i$$

Therefore $\exists c_i \in \mathbb{R},\ x=\sum_{i=0}^{p} c_i B_i \Rightarrow x \in L(B)$

Thus $\forall x, x \in L(A) \Rightarrow x \in L(B)$. So $L(A) \subseteq L(B)$. □

(b) $L(A \cap B) \subseteq L(A) \cap L(B)$.
Proof. $A \cap B \subseteq A$ and $A \cap B \subseteq B$, so $L(A \cap B) \subseteq L(A)$ and $L(A \cap B) \subseteq L(B)$
so $L(A \cap B) \subseteq L(A) \cap L(B)$. □

(c) Give an example in which $L(A\cap B)\neq L(A)\cap L(B)$

Example.

$$A=\{(2,0)\},\ B=\{(1,0)\}.$$
$$A\cap B=\emptyset,\ \text{so } L(A\cap B) \text{ is not defined.}$$
$$L(A)\cap L(B)=\{(c,0): c\in\mathbb{R}\}.$$

1.17: 3. Prove that for any two vectors A and B in $\mathbf{C}^n$, we have the identity

$$\|A+B\|^2=\|A\|^2+\|B\|^2+A\cdot B+\overline{A\cdot B}$$

Proof. $$(A+B)\cdot(A+B)=A\cdot A+B\cdot B+A\cdot B+B\cdot A$$

By the definition of norm:

$$\|A+B\|^2=(A+B)\cdot(A+B)=A\cdot A+B\cdot B+A\cdot B+B\cdot A=\|A\|^2+\|B\|^2+A\cdot B+B\cdot A$$

By theorem 1.11a: $B\cdot A=\overline{A\cdot B}$

Therefore: $\|A+B\|^2=\|A\|^2+\|B\|^2+A\cdot B+\overline{A\cdot B}$ □

3.5: 2. Determine whether the given set is a linear space: All vectors (x,y,z) in $\mathbf{R}^3$ with $x = 0$ or $y = 0$.

Counterexample. Let A = (1, 0, 0). A is in the set since $y = 0$. Let B = (0, 1, 0). B is in the set since $x = 0$. A + B = (1, 1, 0) is not in the set since neither x nor y equals 0. Set is not a linear space because it violates Axiom 1 (closure under addition).

3.5: 10. Determine whether the given set is a linear space: All rational functions f/g with the degree of $f \leq$ the degree of g (including $f = 0$).

The set is a linear space. Let f/g, f_1/g_1, f_2/g_2, f_3/g_3 be functions in the set, and let deg(f) denote the degree of f. By axiomatic definition of a linear space:
(1) Closure under addition.

$$\frac{f_1}{g_1}+\frac{f_2}{g_2}=\frac{g_2 f_1+g_1 f_2}{g_1 g_2}\ .$$

$\deg(g_2f_1 + g_1f_2) = \mathrm{Max}[\deg(g_1f_2), \deg(g_2f_1)]$
$\deg(g_1g_2) = \deg(g_1)+\deg(g_2) \geq \deg(g_1)+\deg(f_2) = \deg(g_1f_2)$ because $\deg(g_2) \geq \deg(f_2)$
$\deg(g_1g_2) = \deg(g_1)+\deg(g_2) \geq \deg(f_1)+\deg(g_2) = \deg(f_1g_2)$ because $\deg(g_1) \geq \deg(f_1)$
Therefore, $\deg(g_1g_2) \geq \deg(g_2f_1 + g_1f_2)$, and the sum of the two functions is a member of the set. Therefore the set is closed under addition.

(2) Closure under multiplication by real numbers.

$\forall c \in \mathbb{R}$, cf/g is still in the set as its degree is unchanged.

(3) Commutative law.

$$\frac{f_1}{g_1}+\frac{f_2}{g_2}=\frac{f_2}{g_2}+\frac{f_1}{g_1}$$

(4) Associative law.

$$\left(\frac{f_1}{g_1}+\frac{f_2}{g_2}\right)+\frac{f_3}{g_3}=\frac{f_1}{g_1}+\left(\frac{f_2}{g_2}+\frac{f_3}{g_3}\right)$$

(5) Existence of zero element.

$$\frac{f}{g}+0=\frac{f}{g}$$

(6) Existence of negatives.

$$\frac{f}{g}+(-1)\frac{f}{g}=0$$

(7) Associative law.

$$a\left(b\frac{f}{g}\right)=(ab)\frac{f}{g}, \forall a, b \in \mathbb{R}$$

(8) Distributive law for addition in the set.

$$a\left(\frac{f1}{g1}+\frac{f2}{g2}\right)=a\frac{f1}{g1}+a\frac{f2}{g2}, \forall a \in \mathbb{R}$$

(9) Distributive law for addition of numbers.

$$(a+b)\frac{f_1}{g_1}=a\frac{f_1}{g_1}+b\frac{f_1}{g_1}, \forall a, b \in \mathbb{R}$$

(10) Existence of identity.

$$1\frac{f}{g}=\frac{f}{g}$$