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Non-integrable fermionic chains near criticality

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Abstract – We compute the Drude weight and the critical exponents as functions of the density in non-integrable generalizations of *XXZ* or Hubbard chains, in the critical high- or low-density regime where the dispersion becomes almost quadratic, the Luttinger liquid description breaks down and the Bethe ansatz cannot be used. Even in the regions where irrelevant terms dominate, no difference between integrable and non-integrable models appears in exponents and conductivity at zero temperature. Our results are based on a fully rigorous two-regime multiscale analysis and a recently introduced partially solvable model.

Introduction. – Understanding whether the behavior of exactly solvable models is generic and persists in the presence of integrability-breaking terms is a central issue in physics. Interacting fermionic chains provide an ideal arena, thanks to the presence of Bethe ansatz solvable models, like the *XXZ* or the Hubbard model, and the fact that cold atoms allow, at least in principle, an experimental verification [1–3].

Exact solutions provide a rather complete picture, including critical exponents and Drude weights at zero temperature for all densities, see, *e.g.*, [4] for a review. The zero-temperature Drude weight (whose finiteness signals an infinite conductivity) for the *XXZ* chain was obtained in [5]. For the *XXZ* chain with non-zero magnetic field a strict lower bound on the Drude weight at finite T was proven in [6] and in [7–10] it was extended to zero field, using the construction of quasi-local charges in [8]. The bounds are derived analytically for infinite temperature (but the non-zero bound is believed to be true for all T) and for special values of $\Delta = \cos n\pi/m$, with n, m coprime (some arguments have been given to show that the Drude weight could be vanishing without this condition). Analogous results are known for the Hubbard model. Dynamical correlations in solvable models are obtained in [11–13]. The Drude weights can be also obtained via dynamical evolution of partitioned systems [14–22].

The above results rely on integrability which is generically lost in realistic systems. Luttinger liquid theory [23] predicts the behavior of the Luttinger model to be generic for non-integrable systems [24]. This was rigorously proved [25] for static zero-temperature properties around the half-filled band case, where the dispersion relation is essentially linear. These limitations are necessary; solvable models show that non-linear dispersion relations produce behaviors different from that of the Luttinger model in the dynamical correlations or at finite temperature; the same is true for static zero-temperature properties at low or high densities.

For the same non-linear lattice dispersion relation, integrable or non-integrable interactions differ by irrelevant terms, usually neglected in the field theoretic Renormalization Group (RG) analysis; for instance, the addition of a next-to-nearest-neighbor interaction makes the *XXZ* model not solvable. The RG irrelevance of these terms does not make them unimportant. On the contrary, at positive temperature the Drude weight depends dramatically on the integrability of the interaction [6,8], in analogy with the classical case [26,27], see also [28–33]. More generally, irrelevant terms are known to play a crucial role for transport properties. For instance, they ensure the cancellation of all the interaction corrections of the optical conductivity of graphene [34,35].

The natural question we address here is the following: for which properties is the behavior found in Bethe ansatz solvable models at $T = 0$ generic even when the Luttinger description breaks down and physics is dominated by irrelevant terms, that is in high- or low-density regime? We answer this question in the case of static zero-temperature properties, away from the Luttinger linear behavior. This is achieved via a two-regime non-perturbative RG scheme

that takes fully into account irrelevant terms. In the second regime, in the spinful case we exploit emerging symmetries by using a recently introduced QFT model [36] with a RG flow exponentially close to the flow of the non-integrable chains. This QFT model is partially solvable in the sense that only the density correlations can be obtained in closed form.

We find that the critical exponents, in the low- or high-density limit, tend to their non-interacting value in the spinless case, while in the spinful case their limiting value depends strongly on the interaction. In both cases the Drude weight behaves essentially as in the non-interacting case. In the special case of solvable interactions, Bethe ansatz results are recovered.

Our analysis shows that there is no qualitative difference between integrable and non-integrable models at zero temperature; the exponents have a similar behavior, the conductivity is infinite and the limiting value of the Drude weight is the same. This is true notwithstanding the fact that in the low- or high-density regime the physics is completely dominated by irrelevant terms; the behavior found in Bethe ansatz solvable models is generic at $T = 0$ even when the Luttinger description breaks down.

Main results. – We consider a model of interacting fermions with Hamiltonian

$$H = -\frac{1}{2}\sum_{x,\,\sigma}(a^+_{x,\,\sigma}a^-_{x+1,\,\sigma} + \text{c.c.}) - \mu\sum_{x,\,\sigma} a^+_{x,\,\sigma}a^-_{x,\,\sigma} + \lambda\sum_{\substack{x,\,y\\ \sigma,\,\sigma'}} w(x-y)a^+_{x,\,\sigma}a^-_{x,\,\sigma}a^+_{y,\,\sigma'}a^-_{y,\,\sigma'}, \tag{1}$$

where $a^\pm_{x,\,\sigma}$ are fermionic creation or annihilation operators, σ is the spin ($\sigma = 0$ in the spinless case and $\sigma = \uparrow, \downarrow$ in the spinning case), x are points on a one-dimensional lattice and $w(x)$ is a short-range potential such that $\sum_x |x|^\alpha |w(x)| < \infty$ for some $\alpha > 0$. In the spinless case with $w(x-y) = \delta_{x,\,y+1}$ the system reduces to the XXZ model and in the spinning case with $\lambda w(x-y) = U\delta_{x,\,y}$ it reduces to the Hubbard model. For other choices of the interaction no solution is known.

The truncated Euclidean correlations are

$$\langle O_{\mathbf{x}_1} \dots O_{\mathbf{x}_n}\rangle = \langle \mathbf{T}(O_{\mathbf{x}_1} \dots O_{\mathbf{x}_n})\rangle_T,$$

where $\mathbf{T}$ is the time ordering operator, $\mathbf{x} = (x_0, x)$, $O_{\mathbf{x}} = e^{Hx_0}O_x e^{-Hx_0}$ and $\langle\cdot\rangle_T$ are the thermodynamic truncated averages. Finally $S(\mathbf{x}-\mathbf{y}) = \langle a^-_{\mathbf{x}} a^+_{\mathbf{y}}\rangle$ denotes the 2-point correlation function.

The density is $\rho_x = \sum_\sigma a^+_{x,\,\sigma}a^-_{x,\,\sigma}$ and the current is defined via the continuity equation that gives

$$j_x = \frac{1}{2i}\sum_\sigma (a^+_{x+1,\,\sigma}a^-_{x,\,\sigma} + a^+_{x,\,\sigma}a^-_{x+1,\,\sigma}).$$

Writing $\mathbf{p} = (p_0, p)$, the (Euclidean) zero-temperature Drude weight D and the susceptibility κ are given by

$$\kappa = \lim_{p\to 0}\lim_{p_0\to 0}\langle\hat\rho_{\mathbf{p}}\hat\rho_{-\mathbf{p}}\rangle_T \quad \text{and} \quad D = \lim_{p_0\to 0}\lim_{p\to 0} D(\mathbf{p})$$

with $D(\mathbf{p}) = \langle\hat j_{\mathbf{p}}\hat j_{-\mathbf{p}}\rangle_T + \Delta$ and

$$\Delta = -\frac{1}{2}\sum_\sigma \langle a^+_{x,\,\sigma}a^-_{x+1,\,\sigma} + a^+_{x+1,\,\sigma}a^-_{x,\,\sigma}\rangle\,.$$

Here $\hat f(\mathbf{p})$ represents the Fourier transform of $f(\mathbf{x})$. A Ward Identity (WI) gives $p_0^2\langle\hat\rho_{\mathbf{p}}\hat\rho_{-\mathbf{p}}\rangle = 4\sin^2 p/2D(\mathbf{p})$ which implies that

$$\lim_{p\to 0}\lim_{p_0\to 0} D(\mathbf{p}) = \lim_{p_0\to 0}\lim_{p\to 0}\langle\hat\rho_{\mathbf{p}}\hat\rho_{-\mathbf{p}}\rangle = 0\,.$$

Note that $D(\mathbf{p})$ is not continuous at $\mathbf{p} = 0$ and it is essential to take the limits in the correct order. Moreover, the limit $p_0 \to 0$ should be taken along the imaginary axis, but the Wick rotation holds for this model [37].

In the non-interacting case the dispersion relation is $-\cos k$ and the Fermi momentum p_F is defined by $\mu = -\cos p_F$. We are interested in the low- or high-density regime, that is for Fermi momentum close to $p_F = 0$ (empty band) or $p_F = \pi$ (filled band) where the dispersion relation is almost quadratic. In the presence of interaction, at fixed μ, the Fermi momentum becomes interaction dependent; we find more convenient to fix the Fermi momentum to be given by $-\cos p_F = \pm 1 \mp r \equiv \mu_R$, with r small and $\mu_R = 1$ in the empty band case while $\mu_R = -1$ in the filled band case. This is achieved by choosing an interaction-dependent chemical potential $\mu = \mu_R + \nu$ with $\nu = \nu(\lambda)$ suitably chosen. The values of the chemical potential corresponding to $p_F = 0, \pi$ are called $\mu_c = \pm 1 + \nu$.

Theorem. *Consider the Hamiltonian (1) with $\mu = \mu_R + \nu(\lambda, r)$ and $\mu_R = -\cos p_F = \pm 1 \mp r$. Then we have*

$$D = \frac{Kv}{\pi} \quad \textit{and} \quad \kappa = \frac{K}{\pi v},$$

where:
– in the spinless case for $|\lambda|$ small we have

$$\nu(\lambda, r) = 2\lambda\hat w(0)\frac{p_F}{\pi} + O(\lambda r),$$

while

$$K = \frac{1-\tau}{1+\tau}\,,$$
$$v = \sin p_F(1 + O(\lambda r^\vartheta))\,,$$
$$\tau = \lambda\frac{\hat w(0) - \hat w(2p_F)}{2\pi v} + O(\lambda^2 r^\vartheta)\,,$$

with $\vartheta \in (1/3, 1/2)$;
– in the spinful case for $\tilde\lambda = \frac{\lambda}{\sin p_F} \geq 0$ small we have $\nu(\lambda, r) = O(\tilde\lambda\sqrt{r})$, while

$$K = \sqrt{\frac{(1-2\nu_\rho)^2 - \nu_4^2}{(1+2\nu_\rho)^2 - \nu_4^2}} \quad v^2 = \bar v^2\frac{(1+\nu_4)^2 - 4\nu_\rho^2}{(1-\nu_4)^2 - 4\nu_\rho^2},$$

where

$$\bar v = \sin p_F(1 + O(\tilde\lambda r^\vartheta) + O(\tilde\lambda^2))\,,$$
$$\nu_4 = \tilde\lambda\frac{\hat w(0)}{2\pi} + O(\tilde\lambda^2)\,,$$
$$\nu_\rho = \frac{\tilde\lambda}{2\pi}\left(\hat w(0) - \frac{\hat w(2p_F)}{2}\right) + O(\tilde\lambda^2)\,.$$

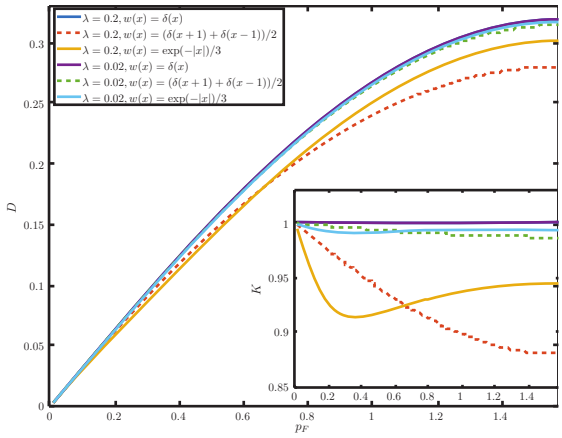

Fig. 1: The main graph is the Drude weight D at fixed λ in the spinless case. The inset shows K.

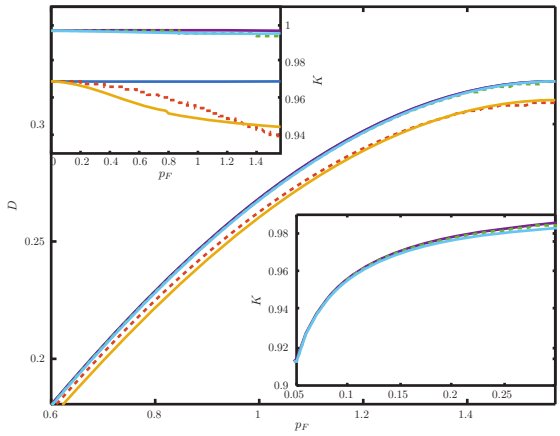

Fig. 2: The main graph is the Drude peak D at fixed λ for the spinful case. The upper inset shows K at fixed $\tilde{\lambda}$ while the lower inset shows K as a function of λ. Colors and dashes are as in fig. 1.

In both cases, $S(\mathbf{x}-\mathbf{y})$ decays for large distance as $|\mathbf{x}|^{1+\eta}$ with $2\eta = K + K^{-1} - 2$.

In the Theorem r is a parameter that measures the distance of μ from the critical chemical potential μ_c. In the spinless case μ_c is shifted by the interaction and we get $\mu_c = 1 + 2\lambda\hat{w}(0)$ for $\mu_R = 1$ and $\mu_c = -1$ for $\mu_R = -1$. In the XXZ chain $h_c + \lambda = \mu_c$. When $r \to 0$ we get $K \to 1$ and $D/\sin p_F \to \frac{1}{\pi}$, that is the critical exponent and the Drude weight tend to their non-interacting values. Figure 1 shows the behavior of D and K as a function of the density close to the critical point; in the XXZ case it closely reproduces the features found by the exact solution, see, *e.g.*, fig. 1 in [38] or fig. 1 in [39].

In the spinful case we rescale the interaction as $\lambda = \tilde{\lambda}\sin p_F$. In terms of $\tilde{\lambda}$ our results hold uniformly in r. In contrast with the spinless case, the theory is strongly interacting since at criticality we have $K \to 1 - \tilde{\lambda}\hat{w}(0)/\pi + O(\tilde{\lambda}^2)$. A remarkable cancellation takes place in the Drude weight and D behaves as in the non-interacting case when $r \to 0$ (at least up to $O(\tilde{\lambda}^2)$ terms), that is

$$\frac{D\pi}{\bar{v}} = \frac{1 + \nu_4 - 2\nu_\rho}{1 - \nu_4 + 2\nu_\rho} \sim 1$$

for $r \sim 0$. Such a behavior is present in the Hubbard model, but it is proven here to be a generic feature. It was missed in previous attempts based of field theoretic RG methods. Figure 2 shows the behavior of D and K for integrable and non-integrable interactions, as a function of λ and $\tilde{\lambda}$. In the Hubbard case fig. 2 reproduces Bethe ansatz result (*e.g.*, figs. 9.2, 9.3 of [4] or figs. 13, 14 of [40]).

RG analysis: the quadratic regime. – We write the Euclidean correlations in terms of a Grassmann integral

$$e^{W(A,\phi)} = \int P(da) e^{-\mathcal{V} - \nu N + B(A,\phi)},$$

where $P(da)$ is a Grassmann integration on the Grassmann algebra generated by the variables $a^{\pm}_{\mathbf{x},\sigma}$ with propagator

$$g(\mathbf{x}-\mathbf{y}) = \frac{1}{4\pi^2}\int e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\hat{g}(\mathbf{k})\,\mathrm{d}\mathbf{k},$$

where

$$\hat{g}(\mathbf{k}) = \frac{1}{-ik_0 - \cos k + \cos p_F}\,.$$

Moreover $\mathcal{V}$ is the interaction and $\nu N = \nu\int \mathrm{d}\mathbf{x}\, a^+_{\mathbf{x},\sigma}a^-_{\mathbf{x},\sigma}$ is a counterterm introduced to take into account the renormalization of the chemical potential, that is we write $\mu = \mu_R + \nu$ with $\mu_R \equiv \cos p_F$. Finally $B(A,\phi)$ is a source term. Differentiating $W(A,\phi)$ with respect to ϕ produces correlations of fermionic fields, while differentiating with respect to A produces correlations of currents or densities.

The starting point of the RG analysis is the decomposition

$$\hat{g}(\mathbf{k}) = \sum_{h=-\infty}^{1}\hat{f}_h(\mathbf{k})\hat{g}(\mathbf{k}) = \sum_{h=-\infty}^{1}\hat{g}^{(h)}(\mathbf{k}), \qquad (2)$$

where $\hat{f}_h(\mathbf{k})$ is a compact support function which is non-vanishing only for $\sqrt{k_0^2 + (\cos k - \cos p_F)^2} \sim 2^h$, see fig. 3. From eq. (2) and the properties of Grassmanian integrations we have that we can write $a^{\pm}_{\mathbf{x},\sigma} = \sum_{h=-\infty}^{1} a^{h,\pm}_{\mathbf{x},\sigma}$ with $P(da) = \prod_{h=-\infty}^{1} P(da^h)$. This decomposition naturally leads to identify two regions, separated by the energy scale $2^{h^*} \sim r$; in the region where the energy is greater than r the dispersion relation is essentially quadratic, while for smaller energies it is essentially linear with a slope of $\sin p_F \sim \sqrt{r}$.

In the high-energy region where $h \geq h^*$ the single scale propagator satisfies the scaling relations $g^{(h)}(\mathbf{x}) \sim 2^{h/2}g^{(0)}(2^h x_0, 2^{h/2}x)$ and the scaling dimension is $D^1 = 3/2 - l/4 - m/2$, where l is the number of a fields and m the number of A fields.

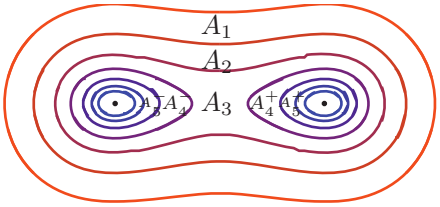

Fig. 3: Schematic representation of the support A_h of the propagator $g^{(h)}(\mathbf{k})$ as a function of h.

We focus on the $A = \phi = 0$ case. We define the effective potential on scale h recursively as

$$V^h(a^{\leq h}) = \log \int P(da^h) e^{V^{h+1}(a^{\leq h+1})},$$

where $a^{\leq h} = \sum_{k=-\infty}^{h} a^k$. It can be written as

$$V^h = \mathcal{L}V^h + \mathcal{R}V^h\,,$$

where $\mathcal{R}V^h$ is sum of all irrelevant terms, that is monomials in the fields with $D^1 < 0$ while

$$\mathcal{L}V^h = 2^{\frac{h}{2}}\lambda_h F_\lambda + 2^h \nu_h \sum_\sigma \int \mathrm{d}\mathbf{x}\, a^{+,\leq h}_{\mathbf{x},\sigma} a^{-,\leq h}_{\mathbf{x},\sigma}$$
$$+ i_h \sum_\sigma \int \mathrm{d}\mathbf{x}\, a^{+,\leq h}_{\mathbf{x},\sigma} \partial_0 a^{-,\leq h}_{\mathbf{x},\sigma} + \delta_h \sum_\sigma \int \mathrm{d}\mathbf{x}\, a^{+,\leq h}_{\mathbf{x},\sigma} \partial^2 a^{-,\leq h}_{\mathbf{x},\sigma} \tag{3}$$

where

$$F_\lambda = \int \mathrm{d}\mathbf{x}\, a^{+,\leq h}_{\mathbf{x},\uparrow} a^{-,\leq h}_{\mathbf{x},\uparrow} a^{+,\leq h}_{\mathbf{x},\downarrow} a^{-,\leq h}_{\mathbf{x},\downarrow}$$

in the spinful case and $F_\lambda = 0$ if the fermions are spinless. Notice the absence of the term $\int a^+_{\mathbf{x},\sigma} \partial a^-_{\mathbf{x},\sigma} \mathrm{d}\mathbf{x}$ and of local terms with six fields due to parity and the Pauli principle, respectively.

After integrating the field a^h we obtain V^{h-1} as a sum of monomials in the fields, that is

$$V^{h-1}(a^{\leq h-1}) = \int W_l^{h-1}(\mathbf{x}_1, \dots, \mathbf{x}_l) \prod_{i=1}^{l} a^{\leq h-1}_{\mathbf{x}_i},$$

where W_l^{h-1} is expressed as a series in the running coupling constant (r.c.c.) $\boldsymbol{\eta}_h = (\nu_k, i_k, \delta_k, \lambda_k)$ (with $\lambda_k \equiv 0$ in the spinless case), $k \geq h$. We can now write $V^{h-1} = \mathcal{L}V^{h-1} + \mathcal{R}V^{h-1}$ as in (3) with $h-1$ replacing h and use the local terms to compute the r.c.c. on scale $h-1$. This produces an expansion of the kernels W_l^h in terms of the r.c.c.. Calling $\epsilon_h = \max_{k>h} |\boldsymbol{\eta}_k|$, we get

$$||W_l^{h-1}|| \leq 2^{h(3/2-l/4)} \sum_n C^n \epsilon_h^n.$$

Convergence in the r.c.c. follows from determinant bounds [41], which imply convergence in λ if the r.c.c. remains close to their initial value during RG iteration.

The above construction gives the recursive relation

$$\boldsymbol{\eta}_{h-1} = \boldsymbol{\eta}_h + \beta^h(\boldsymbol{\eta}_h, \dots, \boldsymbol{\eta}_0)\,.$$

The flow generated by β^h can be analyzed rigorously as in [41]. The main observation is that at $r = 0$ all graphs with a closed fermionic loop vanish while the tadpole graph gives the shift of the chemical potential. Therefore, in the spinless case we get $|i_h|, |\delta_h| \leq Cr^\vartheta |\lambda|$ where the factor r^ϑ is due to the irrelevance of the quartic terms. Similarly the contribution to ν consists of the tadpole graph plus $O(\lambda r)$.

In the spinful case we must also consider λ_h which obeys the recursive relation $\lambda_{h-1} = 2^{\frac{1}{2}}\lambda_h - a\lambda_h^2 + O(\lambda_h^3)$ with $a > 0$, from which $|\lambda_{h^*}| \leq C|\tilde{\lambda}|$. We thus see a non-trivial fixed point that lies outside our convergence radius. For the other r.c.c. we get $|i_{h^*}|, |\delta_{h^*}| \leq Cr^\vartheta|\lambda| + O(\tilde{\lambda}^2)$ while the contribution to ν consists of the tadpole graph plus $O(\lambda r) + O(\tilde{\lambda}^2)$, see also appendix A. This is due to the lack of the dimensional gains of the spinless case for graphs of higher order.

RG analysis: the linear regime. – After the integration of the fields $a^1, a^0, \dots, a^{h^*}$ we arrive at a functional integral of the form $\int P(da^{\leq h^*}) e^{-\mathcal{V}^{h^*}(a)}$, where $P(da^{\leq h^*})$ has a propagator that depends only on the momenta in two disconnected regions around the 2 Fermi points $(0, \pm p_F)$, see fig. 3. Therefore, we write $a^{\leq h^*}$ as the sum of 2 independent fields

$$a^{\leq h^*} = \sum_{\omega=\pm} e^{i\omega p_F x} a^{\leq h^*}_{\omega,\mathbf{x}}$$

with propagator

$$\hat{g}^{(\leq h^*)}_\omega(\mathbf{k}) = \frac{\tilde{f}_{\leq h^*}(\mathbf{k})}{-ik_0 + \omega v_{h^*} k} + \hat{r}^{h^*}(\mathbf{k})\,,$$

where $v_{h^*} = O(\sqrt{r})$, and $\tilde{f}_{\leq h^*}(\mathbf{k})$ is different from 0 only if $k_0^2 + v_{h^*}^2 k^2 \leq 2^{h^*}$. Finally $\hat{r}(\mathbf{k})$ is a bounded correction. In this case the scaling dimension is $D^2 = 2 - l/2$; we write again $V^h = \mathcal{L}V^h + \mathcal{R}V^h$, where $\mathcal{R}V^h$ contains all terms with negative scaling dimension while $\mathcal{L}V^h$ contains ν_h, the renormalization of the chemical potential, and the quartic terms (quadratic marginal terms produce the wave function renormalization Z_h and the renormalized Fermi velocity v_h). In the spinless case the quartic local terms have the form $\lambda_h \int \mathrm{d}\mathbf{x} a^{+,\leq h}_{\mathbf{x},+} a^{-,\leq h}_{\mathbf{x},+} a^{+,\leq h}_{\mathbf{x},-} a^{-,\leq h}_{\mathbf{x},-}$ with

$$\lambda_{h^*} = \lambda(\hat{w}(0) - \hat{w}(2p_F))$$
$$+ \sum_{k=h^*}^{0} (W_4^k(p_F, p_F, -p_F, -p_F) - W_4^k(p_F, -p_F, -p_F, p_F)).$$

Due to the parity of the interaction, the first term is $O(\lambda r)$ while the second is close to $p_F^2 \partial^2 W_4^k$. Since

$$\sum_{k=h^*}^{0} |\partial^2 W_4^k| \leq \sum_{k=h^*}^{0} \lambda^2 2^{h(-1/2+\vartheta)} \leq C\lambda^2 r^{-1/2+\vartheta},$$

we get $\lambda_{h^*} \sim O(\lambda r^{\frac{1}{2}+\vartheta})$, so that it vanishes as $r \to 0$. In the spinful case there are three local quartic terms (if

$p_F \neq \pi/2$):

- $g_{1,h} \int a^+_{\mathbf{x},\omega,\sigma} a^-_{\mathbf{x},-\omega,\sigma} a^+_{\mathbf{x},-\omega,\sigma'} a^-_{\mathbf{x},\omega,\sigma'}$ with $g_{1,h^*} = 2^{h^*/2}(2\tilde\lambda \hat w(2p_F) + O(\tilde\lambda^2))$, where the $2^{h^*/2}$ comes from the scaling dimension;
- $g_{2,h} \int a^+_{\mathbf{x},\omega,\sigma} a^-_{\mathbf{x},\omega,\sigma} a^+_{\mathbf{x},-\omega,\sigma'} a^-_{\mathbf{x},-\omega,\sigma'}$ with $g_{2,h^*} = 2^{h^*/2}(2\tilde\lambda w(0) + O(\tilde\lambda^2))$;
- $g_{4,h} \int a^+_{\mathbf{x},\omega,\sigma} a^-_{\mathbf{x},\omega,\sigma} a^+_{\mathbf{x},\omega,\sigma'} a^-_{\mathbf{x},\omega,\sigma'}$ with $g_{4,h^*} = 2^{h^*/2}(2\tilde\lambda w(0) + O(\tilde\lambda^2))$.

The integration over the time variables produces a factor v^{-n+1} which is compensated by the v^n of the coupling, so that the convergence radius (in λ for the spinless case or $\tilde\lambda$ for the spinful case) is r independent. Observe that the small factor in the effective coupling is produced essentially by the Pauli principle in the spinless case, while it follows from our choice $\lambda = \tilde\lambda \sin p_F$ in the spinful case.

Finally we have to discuss the flow of the running coupling constants. The single scale propagator $\hat g^h(\mathbf{k})$ is the sum of a "relativistic" part

$$\hat g^h_{\omega,rel}(\mathbf{k}) = \frac{1}{Z_h} \frac{\tilde f_h(\mathbf{k})}{-ik_0 + \omega v_h k}$$

and a correction $\hat r^h(\mathbf{k})$, smaller by a factor $\frac{2^h}{v^2_{h^*}}$, that takes into account the non-linear corrections to the dispersion relation. In the spinless case the beta functions for λ_h and v_h are asymptotically vanishing (*i.e.*, the only contributions come from the corrections $\hat r^h$), while

$$|\beta^h_\lambda| \le C \frac{\lambda_h^2}{v_{h^*}} \frac{2^h}{v^2_{h^*}} \quad \text{and} \quad |\beta^h_\delta| \le C\lambda_h \frac{2^h}{v^2_{h^*}}\,.$$

Thus, we get $|\lambda_h| \le C\lambda r^{1/2+\vartheta}$, while $v_{-\infty} = \sin p_F(1 + O(\lambda r^\vartheta))$. Finally we have $Z_h \sim Z_{h^*} 2^{-\eta h}$ with $\eta = \eta_i(\frac{\lambda_{-\infty}}{v_{-\infty}})$, see also appendix B.

In the spinful case if $\lambda > 0$, we get $g_{2,h} \to g_{2,-\infty}$ and $g_{4,h} \to g_{4,-\infty}$ with $g_{2,-\infty} = g_{2,h^*} - g_{1,h^*}/2 + O(\tilde\lambda^2 r^{1/2})$ and $g_{4,-\infty} = g_{4,h^*} + O(\tilde\lambda^2 r^{1/2})$. Finally we have $g_{1,h} \sim \frac{g_{1,h^*}}{1 - a g_{1,h^*}(h-h^*)} \to 0$ as $h \to -\infty$. Similarly we get $\bar v = \sin p_F(1 + O(\tilde\lambda r^\vartheta) + O(\tilde\lambda^2))$.

Emerging chiral model. – Here we focus on the spinful case, since the spinless one is a special case of the following discussion. In the second regime a description of relativistic chiral fermions emerges, up to irrelevant terms, and one needs to exploit its symmetries. A way to do that is to introduce a reference model whose parameters can be fine tuned so that the difference between the running coupling constants of the non-integrable chain and those of the reference model is small. The somewhat natural choice of the Luttinger model does not work, as the difference produced by the g_1 coupling vanishes in a non-summable way.

We introduce a model [36] of fermions $\psi^\pm_{\omega,\sigma}\omega = \pm$ with propagator

$$\hat g^h_{\omega,chi}(\mathbf{k}) = \frac{1}{Z} \frac{\tilde f_{\le N}(\mathbf{k})}{-ik_0 + \omega v k}$$

and interaction given by $\mathcal{V} = \bar g_1 F_1 + \bar g_2 F_2 + \bar g_4 F_4$, where

$$F_1 = \frac{1}{2} \sum_{\omega,\sigma,\sigma'} \int \tilde w(\mathbf{x}-\mathbf{y}) \psi^+_{\mathbf{x},\omega,\sigma} \psi^-_{\mathbf{x},\omega,\sigma'} \psi^-_{\mathbf{y},-\omega,\sigma} \psi^+_{\mathbf{y},-\omega,\sigma'},$$

$$F_2 = \frac{1}{2} \sum_{\omega,\sigma,\sigma'} \int \tilde w(\mathbf{x}-\mathbf{y}) \psi^+_{\mathbf{x},\omega,\sigma} \psi^-_{\mathbf{x},\omega,\sigma} \psi^-_{\mathbf{y},-\omega,\sigma'} \psi^+_{\mathbf{y},-\omega,\sigma'},$$

$$F_4 = \frac{1}{2} \sum_{\omega,\sigma,\sigma'} \int \tilde w(\mathbf{x}-\mathbf{y}) \psi^+_{\mathbf{x},\omega,\sigma} \psi^-_{\mathbf{x},\omega,\sigma} \psi^-_{\mathbf{y},\omega,\sigma'} \psi^+_{\mathbf{y},\omega,\sigma'}\,.$$

Here $\tilde w(\mathbf{x})$ is a short-range interaction, with range r_0 and $\hat w(0) = 1$. Setting

$$\tilde j_{0,\mathbf{x}} = \sum_\omega \tilde\rho_{\omega,\mathbf{x}}\,, \qquad \tilde j_{1,\mathbf{x}} = \sum_\omega \omega \tilde\rho_{\omega,\mathbf{x}}\,,$$

with $\tilde\rho_{\omega,\mathbf{x}} = \sum_\sigma \psi^+_{\omega,\sigma}\psi^-_{\omega,\sigma}$, we get the WI for the fermionic correlations

$$\begin{aligned} &-ip_0 A_0 \langle \hat j_{0,\mathbf{p}} \hat\psi^-_{\mathbf{k}+\mathbf{p},\sigma} \hat\psi^+_{\mathbf{k},\sigma}\rangle_T \\ &+pvA_1 \langle \hat j_{1,\mathbf{p}} \hat\psi^-_{\mathbf{k}+\mathbf{p},\sigma} \hat\psi^+_{\mathbf{k},\sigma}\rangle_T = \\ &\frac{1}{Z}\Big[\langle \hat\psi^-_{\mathbf{k}+\mathbf{p},\sigma} \hat\psi^+_{\mathbf{k}+\mathbf{p},\sigma}\rangle_T - \langle \hat\psi^-_{\mathbf{k},\sigma} \hat\psi^+_{\mathbf{k},\sigma}\rangle_T\Big], \end{aligned} \tag{4}$$

where $A_0 = (1 - \nu_4 - 2\nu_\rho)$, $A_1(1 + \nu_4 - 2\nu_\rho)$, $\nu_4 = \bar g_4/4\pi v$ and $\nu_\rho = (\bar g_2 - \bar g_1/2)/4\pi v$. Similarly, if $\tilde P_\omega = -ip_0 + \omega v p$, the density correlations verify

$$\begin{aligned} &\tilde P_\omega \langle \hat{\tilde\rho}_{\mathbf{p},\omega} \hat{\tilde\rho}_{-\mathbf{p},\omega'}\rangle_T - \nu_4 \tilde P_{-\omega} \langle \hat{\tilde\rho}_{\mathbf{p},\omega} \hat{\tilde\rho}_{-\mathbf{p},\omega'}\rangle_T \\ &-2\nu_\rho \tilde P_{-\omega} \langle \hat{\tilde\rho}_{\mathbf{p},-\omega} \hat{\tilde\rho}_{-\mathbf{p},\omega'}\rangle_T = -\delta_{\omega,\omega'} \frac{\tilde P_{-\omega}}{2\pi Z^2}. \end{aligned} \tag{5}$$

Note in the above WI the presence of the *anomalies*, that is the terms in ν_ρ and ν_4, which are linear in the couplings $\bar g_i$. The model differs from the Luttinger model for the presence of the $\bar g_1$ term; it is however defined so that it is invariant under the chiral phase transformation

$$\psi^\pm_{\mathbf{x},\omega,\sigma} \to e^{\pm i\alpha_{\mathbf{x},\omega}} \psi^\pm_{\mathbf{x},\omega,\sigma}$$

which implies, thanks to (5), that the density correlations can be explicitly computed even if the model is not solvable, see [25,36]. We choose $\tilde w$ of the form $\tilde w(\mathbf{x}) = \bar w(x^2 + x_0^2/v^2)$, where $\bar w$ has range $r_0 = 2^{-h^*}$ and satisfies $\int d\mathbf{x} |\tilde w(\mathbf{x})| = 1$. It acts as an ultraviolet cut-off that allows us to integrate safely the scales $h \ge h^*$ and arrives at an effective potential $\overline{V}^{h^*}$, differing from V^{h^*} discussed in the previous section by irrelevant terms. We can choose the bare parameters $\bar g_i, v$ of the reference model so that its running coupling constants differ from those of model (1) by exponentially decaying terms $O(2^{\vartheta h})$ and the ratio of the Z tends to 1; this is achieved by choosing $\bar g_i = g_{i,h^*} + O(\sqrt{r}\tilde\lambda^2))$ and $v = \sin p_F(1 + O(\tilde\lambda r^\vartheta) + O(\tilde\lambda^2))$. This implies that

$$D(\mathbf{p}) = \frac{Z_1^2}{Z^2} \langle \hat j_{1,\mathbf{p}} \hat j_{1,-\mathbf{p}}\rangle_T + R_0(\mathbf{p}), \tag{6}$$

where Z_1 is the current wave function normalization and $R_0(\mathbf{p})$ is a *continuous* function in $\mathbf{p}$ (in contrast

with the first addend in the r.h.s.); we use the WI $\lim_{p\to 0}\lim_{p_0\to 0} D(\mathbf{p}) = 0$ to fix $R_0(0)$ so that we get

$$D(\mathbf{p}) = \frac{Z_1^2}{\pi Z^2 v v_1}\frac{[(1+\nu_4+2\nu_\rho)+v_2^2(1-\nu_4-2\nu_\rho)]p_0^2}{p_0^2+v_2^2v^2p^2}$$

with $v_2^2 = \frac{(1+\nu_4)^2-4\nu_\rho^2}{(1-\nu_4)^2-4\nu_\rho^2}$, $v_1 = (1+\nu_4)^2-4\nu_\rho^2$. The identity

$$\langle j_{\mathbf{p}} a^-_{\mathbf{k}+\mathbf{p}_F,\sigma} a^+_{\mathbf{k}+\mathbf{p}+\mathbf{p}_F,\sigma}\rangle_T = Z_1\langle \tilde{j}_{1,\mathbf{p}}\psi^-_{\mathbf{k}+\mathbf{p},\sigma}\psi^+_{\mathbf{k},\sigma}\rangle_T$$

allows us to fix Z_1, Z; indeed comparing (4) with the WI for the chain

$$\begin{aligned}&-ip_0\langle\hat\rho_{\mathbf{p}}\hat a^-_{\mathbf{k},\sigma}a^+_{\mathbf{k}+\mathbf{p},\sigma}\rangle_T + p\langle\hat j_{\mathbf{p}}\hat a^-_{\mathbf{k},\sigma}a^+_{\mathbf{k}+\mathbf{p},\sigma}\rangle_T = \\ &\langle\hat a^-_{\mathbf{k},\sigma}a^+_{\mathbf{k},\sigma}\rangle_T - \langle\hat a^-_{\mathbf{k}+\mathbf{p},\sigma}a^+_{\mathbf{k}+\mathbf{p},\sigma}\rangle_T \end{aligned}\tag{7}$$

we get the consistency relation

$$\frac{Z_1}{Z} = v(1+\nu_4-2\nu_\rho).$$

Proceeding in a similar way for the susceptibility we obtain the expressions in the Theorem.

Conclusions. – We analyze non-integrable generalizations of the Hubbard chain in the low- and high-density regimes where the Luttinger description breaks down. Our methods are based on a multiscale decomposition of the propagator of the theory and are able to take into account, in a rigorous and quantitative way, the irrelevant terms normally neglected in the RG analysis. Our main conclusion is that no qualitative difference between solvable and non-solvable models is seen in exponents and conductivity at zero temperature, even in regions where the Luttinger liquid description is not valid and the physics is completely dominated by irrelevant terms. Both for integrable and non-integrables models, the anomalous critical exponents vanish or do not depend on the spinless or spinful nature of fermions, while the Drude weight tends to the same non-interacting values independently of the presence of spin. Such results have been established by rigorous RG methods able to take into account irrelevant terms. The possibility of extending such methods to deal with the issue of finite-temperature Drude weight in non-integrable models is an important open question.

Appendix A: flow of the running coupling constants in the quadratic regime. – We give some extra detail on the flow of the r.c.c. in the quadratic regime. Note that at $r = 0$ and $T = 0$ we have

– empty band case: $p_F = 0$, $e(k) = -\cos k + 1$, and

$$g(\mathbf{x}) = \chi(x_0 > 0)\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}e^{-ikx-e(k)x_0};$$

– filled band case: $p_F = \pi$, $e(k) = -\cos k - 1$, and

$$g(\mathbf{x}) = -\chi(x_0 \le 0)\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}e^{-ikx-e(k)x_0}.$$

Therefore, all the graphs with order greater than 1 with two external lines are vanishing if computed at the Fermi points and $r = 0$. Indeed all one-particle reducible graphs are vanishing due to the support properties of the propagator. This implies that there is always a closed fermionic loop which vanishes as the propagator is proportional to $\chi(x_0 > 0)$ or $\chi(x_0 \le 0)$. At first order there are two contributions: the tadpole graph at $r = 0$ contributes only to ν and gives $2\lambda\hat w(0)p_F/\pi$ with $p_F = 0, \pi$; the other graph is vanishing for non-local interactions (the local potential does not contribute) since $v(\mathbf{x}-\mathbf{y})g(\mathbf{x}-\mathbf{y})$ is proportional to $v(x,y)\delta_{x,y} = 0$.

The flow equations for i_h, δ_h have the form $i_{h-1} = i_h + \beta^h_i$, $\delta_{h-1} = \delta_h + \beta^h_i$. In the spinless case the fact that there are no quartic running coupling constants produces an improvement of $O(2^{h\vartheta})$ with respect to the dimensional bound. As we noticed above all the contributions with two external lines computed at the Fermi points are vanishing for $r = 0$, except for the tadpole which contributes only to ν_h. There is therefore a gain $r2^{-h}$ in the beta function for z, δ, and a further gain $2^{h\vartheta}$ (due to the irrelevance of the quartic terms if the order is greater then 1 and to the fact that the derivative can be applied on the interaction at first order), so we get $|i_h|, |\delta_h| \le \sum_{k=h}^1 C|\lambda| r2^{-k}2^{k\vartheta}$ and finally $z_{h^*}, \delta_{h^*} = O(\lambda r^\vartheta)$. The same argument can be used for the renormalization of the chemical potential ν_h and ν_0 is the tadpole plus $\sum_{h=h^*}^1 \lambda 2^h r2^{-h}2^{\vartheta h} = O(\lambda r)$; as a consequence the shift of the critical chemical potential is linear in λ as stated in the Theorem.

In the spinful case, the contributions at first order to the flow of i_h, δ_h give $\tilde\lambda\sum_{h\ge h^*} r2^{-h}2^{\vartheta h} \le Cr^\vartheta\tilde\lambda$ for the same reason as in the spinless case. There is, however, no gain due to the irrelevance of the interaction at larger orders so that they give $\tilde\lambda^2 C\sum_{h\ge h^*} r2^{-h} \le C\tilde\lambda^2$ as the quartic terms are now relevant. Finally, the value of ν is the tadpole plus $\sum_{h=h^*}^1 \tilde\lambda 2^h r2^{-h} = O(\tilde\lambda\sqrt{r})$.

Appendix B: flow of the running coupling constants in the linear regime. – In the spinless case the beta functions for λ_h and v_h are convergent and asymptotically vanishing, $|\beta^h_\lambda| \le C\frac{\lambda_h^2}{v_{h^*}}\frac{2^h}{v^2_{h^*}}$, $|\beta^h_\delta| \le C\frac{\lambda_h^2}{v^2_{h^*}}\frac{2^h}{v^2_{h^*}}$. Assuming inductively that $|\lambda_h| \le C\lambda r^{1/2+\vartheta}$ and using that $\frac{2^h}{v^2_{h^*}} \le 2^{h-h^*}$ one gets that

$$|\lambda_{h-1} - \lambda_{h^*}| \le \sum_{k=h}^{h^*} r^{1+2\vartheta}\frac{\lambda^2}{v_{h^*}}2^{k-h^*} \le C\lambda^2 r^{1/2+\vartheta}\tag{8}$$

and $v_{-\infty} = v_{h^*} + O(\frac{\lambda^2_{h^*}}{v^2_{h^*}}) \sim r^{\frac12}$. Moreover, $\frac{Z_{h-1}}{Z_h} = 1 + \beta^1_z + \beta^2_z$, where β^2 contains the contributions from the irrelevant terms, like the quadratic corrections to the dispersion relation, and is $O(\lambda\frac{\gamma^h}{v_{h^*}})$. Finally at first order δ_h has contributions only from non-local terms, the derivative is applied on the interaction and is bounded by $\lambda/v\sum_{k\le h^*} 2^k$ both in the spinful and in the spinless case.

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