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A dynamic analysis of nash equilibria in search models with fiat money[☆]

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ABSTRACT

We analyze the rise in the acceptability fiat money in a Kiyotaki–Wright economy by developing a method that can determine dynamic Nash equilibria for a class of search models with genuine heterogeneous agents. We also address open issues regarding the stability properties of pure strategy equilibria and the presence of multiple equilibria, numerical experiments illustrate the liquidity conditions that favor the transition from partial to full acceptance of fiat money, and the effects of inflationary shocks on production, liquidity, and trade.

1. Introduction

One central question of monetary economics is how an object that does not bring utility *per se* becomes accepted as a means of payment. It is well understood that the emergence of money depends on both trust and coordination of beliefs. While some recognize this observation and simply assume that money is part of the economic system, others have tried to explain the acceptance of money as the result of individuals' interactions in trade and production activities.[1] Among the best-known attempts to formalize the emergence of money in decentralized exchanges is Kiyotaki and Wright (1989, henceforth, KW). The static analysis in KW provides important insights on how the specialization in production, the technology of matching, and the cost of holding commodities condition the emergence of monetary equilibria. Nevertheless, important issues are yet to be resolved. First, one would like to know if and how convergence to a particular long run equilibrium occurs from an arbitrary initial state of the economy. Historical accounts describe the different patterns that societies have followed in adopting objects as a means of payment.[2] What are the dynamic conditions that lead individuals in a KW economy to accept either commodity or fiat money? Second, static analysis gives little guidance about the short run consequences of a shock that causes, for instance, a sudden rise in inflation. How does the degree of acceptability of commodity and fiat money change with inflation?

The determination of dynamic equilibria in a KW environment is challenging. In an effort to improve its tractability, new classes of monetary search models have been proposed. These have incorporated some features of centralized exchanges but have also eliminated others, most notably the genuine heterogeneity across individuals and goods, and the storability of goods (see Lagos et al. (2017), for a recent review). Restoring these features turns out to be a useful exercise for characterizing the rise of money as a dynamic phenomenon.

The study of money acceptance in a KW environment requires a departure from the conventional set of tools employed to characterize the dynamics of an economy with centralized markets. Our method combines Nash's (1950) definition of equilibrium with Perron's iterative approach to prove the stable manifold theorem (see, among others, Robinson (1995)). This is the first work, to our knowledge, that shows how to determine pure strategy dynamic Nash equilibria in a KW search environment with

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1 Economic textbooks sometimes warn readers that the acceptance of fiat money cannot be taken for granted. For instance, Mankiw (2006, pp. 644–45), observed that in Moscow in the late 1980s, when the Soviet Union was breaking up, some preferred cigarettes to rubles as means of payment.

2 For a classic review of the rise of early means of payments see Quiggin (1949). For the institutional and historical conditions that favored the dissemination of fiat money see Goetzmann (2016).

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fiat money. Previous works on the subject considered economies without fiat money, and often assumed bounded rationality.[3] An exception is Iacopetta (2019) that also studies dynamic Nash equilibria in a KW environment, but does not consider fiat money. The work presented here extends the KW environment of Iacopetta (2019) in that it introduces fiat money and also considers seignorage (Li, 1994, 1995). In the present environment it is possible to explicitly address how the distribution of individuals' characteristics affects the emergence of a partial or full monetary equilibrium.

Steady-state results echo those of the inventory-theoretic models of money (e.g., Baumol (1952), Tobin (1956), and Jovanovic (1982)). For instance, higher levels of seigniorage may induce some to keep commodities in the inventory instead of accepting money, as a way to minimize the odds of being hit by seignorage tax. The dynamic analysis, however, generates novel results: it shows how changes in the liquidity of assets other than money can alter the proportion of individuals who accept fiat money in transactions. For instance, it reveals that an economy that converges to a long run equilibrium in which all prefer fiat money to all types of commodities (full acceptance) may go through a phase in which only a fraction of individuals do so (partial acceptance).[4]

The remainder of the paper is organized as follows: Section 2 describes the economic environment, characterizes the evolution of the distribution of inventories and money and defines a Nash equilibrium. Section 3 overviews steady-state Nash equilibria for some specifications of the model. Section 4 presents a methodology to determine Nash equilibria. Section 5 illustrates the acceptability of money and discusses multiple steady states through numerical experiments. Section 6 contains welfare considerations. Section 7 comments on future research. Appendix A contains proofs and mathematical details that are omitted in the main text. Appendix B explains how the stable manifold theorem is related to our solution algorithm.

2. The model

This section describes the economic environment, characterizes the evolution of the distribution of inventories and money and defines a Nash equilibrium.

2.1. The environment

The model economy is a generalization of that described in KW. There are four main differences. First, to facilitate the analysis of the dynamics, time is continuous. Second, the model is extended to deal with seignorage, following the approach devised by Li (1994, 1995): government agents randomly confiscate balances from money holders and use the proceedings to purchase commodities. Third, as described in Wright (1995) agents are not necessarily equally divided among the three types. Fourth, similar to Lagos et al. (2017), we obtain the type of equilibria that emerge in the Model B of KW by reshuffling the order of the storage costs across the three types of goods, rather than altering the patterns of specialization in production.

The economy is populated by three types of infinitely-lived agents; there are N_i individuals of type i, with $i = 1, 2, 3$, where N_i is a very large number. The total size of the population is $N = N_1 + N_2 + N_3$ and the fraction of each type is denoted with $\theta_i = \frac{N_i}{N}$. A type i agent consumes only good i and can produce only good $i + 1$ (modulo 3). Production occurs immediately after consumption. Agent i's instantaneous utility from consuming a unit of good i and the disutility of producing good $i + 1$ are denoted by U_i and D_i, respectively, with $U_i > D_i > 0$, and their difference with $u_i = U_i - D_i$. The storage cost of good i is c_i, measured in units of utility.[5] In addition to the three types of commodity, there is a fourth object, called money and denoted by m, which does not bring utility *per se*: it only serves as a means of transaction. Fiat money is indivisible. Denoting with M the fraction of the population holding fiat money, the total quantity of fiat money is $Q = NM$. There is no cost for storing money. At each instant in time, an individual can hold one and only one unit of any type i good or one unit of money.[6]

The discount rate is denoted by $\rho > 0$. A pair of agents is randomly and uniformly chosen from the population to meet for a possible trade. The matching process is governed by a Poisson process with exogenous arrival rate $\frac{\alpha N}{2}$, where $\alpha > 0$, i.e., there is a constant return to scale matching technology. Hence, after a pair is formed, the expected waiting time for the next pair to be formed is $\frac{2}{\alpha N}$. A bilateral trade occurs if, and only if, it is mutually agreeable. Agent i always accepts good i but never holds it because, provided that u_i is sufficiently large, there is immediate consumption (see KW, Lemma 1, p. 933). Therefore, agent i enters the market with either one unit of good $i + 1$, or $i + 2$, or with one unit of m.

We introduce seignorage following Li (1994, 1995): The government extracts seignorage revenue from money holders in the form of a money tax — this device has been used by many others, including in the recent work of Deviatov and Wallace (2014). In particular, government agents meet and confiscate money from money holders. The arrival rate of a government agent for a money holder is δ_m. As described in Li (1995), this arrival rate is independent from the matching rate α — government agents are not counted as part of the N traders. A rise in δ_m can be interpreted as the result of an increase of the government agents' confiscation effort. A money holder of type i, whose unit of money is confiscated, returns to the state of production without consumption, produces a new commodity $i+1$, and incurs a disutility D_i. With the proceedings of the tax revenue, the government purchases goods from commodity holders. These encounters are governed by a Poisson process with arrival rate δ_g. The government runs, on average, a balanced budget. This requires that $\delta_m M = \delta_g(1 - M)$, implying that $\delta_g = \frac{\delta_m M}{1-M}$. We allow the government to alter the rate of seignorage δ_m, but, in order to simplify the dynamic analysis, the government does not change the real balances in circulation, i.e., the initial level of fiat money is given and does not change over time. We will compare, however, steady state equilibria of economies with different levels of M.

3 See, for instance, the works of Marimon et al. (1990) and Başçi (1999) with intelligent agents, and of Brown (1996) and Duffy and Ochs (1999, 2002) with controlled laboratory experiments. Matsuyama et al. (1993), Wright (1995), Luo (1999) and Sethi (1999) use evolutionary dynamics. Kehoe et al. (1993) show that mixed strategy equilibria could generate cycles, sunspots, and other non-Markovian equilibria. Renero (1998), however, proves that it is impossible to find an initial condition from which an equilibrium pattern converges to a mixed strategy, steady-state equilibrium. Oberfield and Trachter (2012) find that, in a symmetric environment, as the frequency of search increases, cycles and multiplicity in mixed strategy tend to disappear.

4 Shevchenko and Wright (2004) also study partial acceptability in pure strategies. They focus, however, on steady-state analysis.

5 There is no restriction on the sign of c_i. A negative storage cost is equivalent to a positive return.

6 The assumption that an individual can have either 0 or 1 unit of an asset greatly simplifies the analysis and makes the decision about the acceptance of money more transparent. There has been a significant amount of previous work on asset-holding restrictions. See, among others, Diamond (1982), Rubinstein and Wolinsky (1987), Cavalcanti and Wallace (1999), and Duffie et al. (2005).

2.2. Distribution of commodities and fiat money

Let $p_{i,j}(t)$ denote the proportion of type i agents that hold good j at time t. A type i with good j has to decide during a meeting whether to trade j for k, where $j, k = i+1, i+2$, or m (henceforth we no longer mention the ranges of the indices i, j and k, unless necessary to prevent confusion). Agent i's decision in favor of trading j for k is denoted by $s^i_{j,k} = 1$, and that against it by $s^i_{j,k} = 0$. The evolution of $p_{i,j}$, for a given set of strategies, $s^i_{j,k}(t)$, is governed by a system of differential equations[7] (the time index is dropped):

$$\begin{aligned}\dot{p}_{i,i+1} =& \alpha \left\{ \sum_{i'} \sum_{k} p_{i,k} p_{i',i+1} s^i_{k,i+1} s^{i'}_{i+1,k} + \sum_{i'} p_{i,k} p_{i',i} s^{i'}_{i,k} \right. \\ & \left. - \sum_{i'} \sum_{k} p_{i,i+1} p_{i',k} s^i_{i+1,k} s^{i'}_{k,i+1} \right\} \\ & + \delta_m p_{i,m} - \delta_g p_{i,i+1} \end{aligned} \tag{1}$$

$$\begin{aligned}\dot{p}_{i,i+2} =& \alpha \left\{ \sum_{i'} \sum_{k} p_{i,k} p_{i',i+2} s^i_{k,i+2} s^{i'}_{i+2,k} - \sum_{i'} \sum_{k} p_{i,i+2} p_{i',k} s^i_{i+2,k} \right. \\ & \left. \times\ s^{i'}_{k,i+2} \right\} - \delta_g p_{i,i+2} \end{aligned} \tag{2}$$

$$\begin{aligned}\dot{p}_{i,m} =& \alpha \left\{ \sum_{i'} \sum_{k} p_{i,k} p_{i',m} s^i_{k,m} s^{i'}_{m,k} - \sum_{i'} \sum_{k} p_{i,m} p_{i',k} s^i_{m,k} s^{i'}_{k,m} \right\} \\ & - \delta_m p_{i,m} + \delta_g (p_{i,i+1} + p_{i,i+2}). \end{aligned} \tag{3}$$

Focusing on the top equation ($\dot{p}_{i,i+1}$), the first two sums inside the brackets, starting from the left, account for events that lead to an increase in the share of individuals i with $i+1$. Specifically, the term $p_{i,k}p_{i',i+1}$ is the probability that a type i with good k meets a type i' with good $i+1$, and $s^i_{k,i+1}s^{i'}_{i+1,k}$ calculates their willingness to swap goods: if they both agree to trade, $s^i_{k,i+1}s^{i'}_{i+1,k} = 1$; if one of the two does not, $s^i_{k,i+1}s^{i'}_{i+1,k} = 0$. The term $p_{i,k}p_{i',i}$ considers the residual case in which type i with good k meets a type i' with good i. Because type i always accepts good i, trade take places as long as $s^{i'}_{i,k} = 1$. The third sum accounts for events that cause a decline in $p_{i,i+1}$. Finally, the last two terms, $\delta_m p_{i,m}$ and $\delta_g p_{i,i+1}$, measure the overall amount of fiat money that the government confiscates from money holders ($p_{i,m}$) and the amount of goods $i+1$ it buys from type i agents — δ_m and δ_g are the government's Poisson rates of intervention.

The extended form of Eqs. (1)–(3) consists of nine non-autonomous non-linear differential equations in the nine unknowns $p_{i,j}(t)$, for $i = 1, 2, 3$, and $j = i+1, i+2, m$ (they are non-autonomous equations because $s^i_{j,k}(t)$ depends on t). Nevertheless, because $p_{i,i}(t) = 0$,

$$p_{i,i+1}(t) + p_{i,i+2}(t) + p_{i,m}(t) = \theta_i. \tag{4}$$

Another restriction comes from the following accounting relationship:

$$p_{1,m}(t) + p_{2,m}(t) + p_{3,m}(t) = M. \tag{5}$$

Therefore, in (1)–(3), there are only five independent equations, and the state of the economy can be represented by the five-dimensional vector $\mathbf{p}(t) = (p_{1,2}(t), p_{2,3}(t), p_{3,1}(t), p_{1,m}(t), p_{2,m}(t))$.

[7] We assume that the influence of any particular individual on the system is negligible. Eqs. (1)–(3) should be interpreted as the limit of the stochastic evolution of the inventories of an economy with a finite number of agents. See Araujo (2004) and Araujo et al. (2012) for a discussion of the system properties of similar economies with a finite number of agents.

Let Ω be the set of $p^i_{j,k}(t)$ that satisfies (4) and (5). For any sets of strategies $s^i_{j,k}(t)$, the solution of (1)–(3) maps Ω into itself. Because Ω is compact and convex, the system (1)–(3) admits at least one fixed point for any constant sets of strategies $s^i_{j,k}$. Proposition 3 shows that in the simple scenario with no fiat money ($M = 0$), the fixed point is unique and globally attractive. Section 5 studies the uniqueness and global attractiveness of the fixed point numerically for an economy with fiat money.

2.3. Value functions

The system of equations (1)–(3) specifies the evolution of the economy for a given set of strategies $s^i_{j,k}(t)$. We turn now to the individuals' decisions about trading strategies. The strategies of a particular agent of type i, a_i are denoted with $\sigma^i_{j,k}(t)$. This agent takes for given the strategies of the rest of the population, $s^i_{j,k}(t)$, including agents of her own type, and knows the initial state $\mathbf{p}(0)$. Let $V_{i,j}(t)$ be the integrated expected discounted flow of utility from time t onward of this particular agent a_i with good j at time t. Then,

$$V_{i,j}(t) = \int_t^\infty e^{-\rho(\tau - t)} \sum_l \pi^i_{l,j}(\tau, t) \upsilon_{i,l}(\mathbf{p}(\tau)) d\tau, \tag{6}$$

where $\pi^i_{l,j}(\tau, t)$ is the probability that a_i, who carries good j at time t and plays strategy $\sigma^i_{j,k}(t)$, holds good l at time $\tau \geq t$, and $\upsilon_{i,l}(\mathbf{p})$ is a_i's flow of utility, net of storage costs, associated to the distribution of holdings $\mathbf{p}$. Observe that $\pi^i_{l,j}(\tau, t)$ and $\upsilon_{i,l}(\mathbf{p})$ both depend on what other individuals do, that is, they are affected by $s^{i'}_{j',k'}(\tau)$, $\sigma^i_{j',k'}(\tau)$ — where $i' = 1, 2, 3$, and $j', k' = i+1, i+2, m$. Because $\rho > 0$ and $\upsilon_{i,l}(\mathbf{p})$ is bounded, the integral in (6) is well defined for any $s^i_{j,k}(t)$ and any $\mathbf{p}(t)$. Appendix A contains the expression of $\upsilon_{i,l}(\mathbf{p})$ and the evolution of $\pi^i_{l,j}(\tau, t)$. It also shows the following result about the evolution of $V_{i,j}(t)$.

Proposition 1. *The evolution of $V_{i,j}$ in (6) satisfies (time index is dropped):*

$$(\alpha + \delta_k + \rho) V_{i,j} = \dot{V}_{i,j} + \phi_{i,j}, \tag{7}$$

where $\delta_k = \delta_g$ and

$$\begin{aligned}\phi_{i,j} =& \alpha \left\{ \sum_{i'} \sum_{k \neq i} p_{i',k} \sigma^i_{j,k} s^{i'}_{k,j} V_{i,k} + \sum_{i'} p_{i',i} s^{i'}_{i,j} (V_{i,i+1} + u_i) \right. \\ & \left. + \sum_{i',k} p_{i',k} (1 - \sigma^i_{j,k} s^{i'}_{k,j}) V_{i,j} \right\} \\ & + \delta_g V_{i,m} - c_j, \end{aligned} \tag{8}$$

when $j = i+1$ or $i+2$, and $\delta_k = \delta_m$ and

$$\begin{aligned}\phi_{i,m} =& \alpha \left\{ \sum_{i'} \sum_{k \neq i} p_{i',k} \sigma^i_{m,k} s^{i'}_{k,m} V_{i,k} + \sum_{i'} p_{i',i} s^{i'}_{i,m} (V_{i,i+1} + u_i) \right. \\ & \left. + \sum_{i',k} p_{i',k} (1 - \sigma^i_{m,k} s^{i'}_{k,m}) V_{i,m} \right\} \\ & + \delta_m (V_{i,i+1} - D_i), \end{aligned} \tag{9}$$

when $j = m$.

Proof. See Appendix A. ■

The first sum in (8), counting from the left, is the expected flow of utility of agent a_i with good j conditional on meeting an agent i' who carries a good $k \neq i$. Such a meeting occurs with probability $p_{i',k}$, and trade follows if $\sigma^i_{k,j} s^{i'}_{j,k} = 1$. In such a case, a_i leaves the meeting with good k (i.e., with continuation value $V_{i,k}$). Similarly, the second sum accounts for a_i's expected flow of utility, conditional on meeting an agent i' who carries good i. The last sum refers to meetings in which no trade occurs, in which case a_i is left with good j. The term $\delta_g V_{i,m}$ is the expected continuation value for meeting government agents who buy a_i's good j using fiat money, and c_j is the cost of storage. Because the term $\delta_g V_{i,m}$ appears in (8) for both $j = i+1$ and $j = i+2$, the level of the seignorage does not directly affect the optimal response $\sigma^i_{k,j}(t)$ — it may do so only indirectly through $\mathbf{p}(t)$.

2.4. Best response and Nash equilibrium

Agent a_i's best response to the set of strategies of the other agents, $s^i_{j,k}(t)$, is a set of strategy $\sigma^i_{j,k}(t)$ that maximizes her expected flow of utility in (6):

$$V_{i,j}\left(t, \left\{\sigma^i_{k,l}(\tau)\right\}_{\substack{k,l=i+1,i+2,m\\ \tau>t}}\right) = \sup_{\tilde{\sigma}^i_{k,j}} V_{i,j}\left(t, \left\{\tilde{\sigma}^i_{k,l}(\tau)\right\}_{\substack{k,l=i+1,i+2,m\\ \tau>t}}\right),$$

$$\text{for } \forall t \text{ and } \forall j. \tag{10}$$

A useful characterization of the best response function is the following:

Proposition 2. *Let $\Delta^i_{j,k}(t) \equiv V_{i,j}(t) - V_{i,k}(t)$. The strategy $\sigma^i_{j,k}(t)$ is a best response of a_i to $s^i_{j,k}(t)$, for a given $\mathbf{p}(0) = \mathbf{p}_0$, if and only if*

$$\sigma^i_{j,k}(t) = \begin{cases} 1 & \text{if } \Delta^i_{j,k}(t) < 0 \\ 0 & \text{if } \Delta^i_{j,k}(t) > 0 \\ s & \text{if } \Delta^i_{j,k}(t) = 0\,. \end{cases} \tag{11}$$

where $0 \leq s \leq 1$.

Proof. See Appendix A. ■

In the isolated episodes in which a_i is indifferent in trading j for k, with $j \neq k$, that is, $\Delta^i_{j,k}(t) = 0$, we adopt the tie-breaking rule $\sigma^i_{j,k}(t) = 0.5$.[8] Clearly, (11) implies that $\sigma^i_{j,k} = 1 - \sigma^i_{k,j}$. We also set $\sigma^i_{j,j} = 0$, that is, a_i never trades j for j. Therefore, the strategy set of agent a_i can be represented by $\boldsymbol{\sigma}^i(t) = (\sigma^i_{i+1,m}(t), \sigma^i_{i+2,m}(t), \sigma^i_{i+1,i+2}(t))$, a piecewise continuous function from $\mathbb{R}^+$ into $\Sigma = \{(1,1,1), (1,0,1), (1,1,0), (0,1,0), (0,0,1), (0,0,0)\}$. Although agent a_i has eight possible trading choices at each point in time, a simple transitivity trading rule (for instance, if $\sigma^1_{2,3} = 0$ and $\sigma^1_{2,m} = 1$, then it must also be that $\sigma^1_{3,m} = 1$) reduces her choices to the six contained in Σ — this applies to any type $i = 1, 2, 3$. We call $\boldsymbol{\sigma}(t) = (\boldsymbol{\sigma}^1(t), \boldsymbol{\sigma}^2(t), \boldsymbol{\sigma}^3(t)) \in \Sigma^3$ the best responses of the three particular agents a_i. Similarly, $\mathbf{s}(t) = (\mathbf{s}^1(t), \mathbf{s}^2(t), \mathbf{s}^3(t)) \in \Sigma^3$ denotes agents' symmetric strategies, with $\mathbf{s}^i(t) = (s^i_{i+1,m}(t), s^i_{i+2,m}(t), s^i_{i+1,i+2}(t)) \in \Sigma$. Next, following Nash (1950), we define an equilibrium by means of a function $\boldsymbol{\sigma} = \mathcal{B}(\mathbf{s})$ that associates the best response $\boldsymbol{\sigma}(t)$ to a set of strategies $\mathbf{s}(t)$. The function $\mathcal{B}$ transforms a piecewise continuous function $\mathbf{s} : \mathbb{R}^+ \to \Sigma^3$ into another piecewise continuous function $\boldsymbol{\sigma} = \mathcal{B}(\mathbf{s}) : \mathbb{R}^+ \to \Sigma^3$.

Definition 1 (*Nash Equilibrium*). Given an initial distribution $\mathbf{p}_0$, a set of strategies $\mathbf{s}^*$ is a Nash equilibrium if it is a fixed point of the map $\mathcal{B}$ [9]:

$$\mathbf{s}^* = \mathcal{B}(\mathbf{s}^*). \tag{12}$$

This definition equilibrium requires, therefore, that $\boldsymbol{\sigma}$, the best response to the set of strategies $\mathbf{s}$, be equal to $\mathbf{s}$.

A general proof of the existence of such an equilibrium, for any given $\mathbf{p}_0$, cannot be obtained with the standard fixed-point argument based on Kakutani or Brouwer theorems applied to finite games. These theorems would require the best response function to be a continuous map on a convex and compact set. Compactness, however, cannot be verified in our infinite time horizon set up. Nevertheless, Proposition 4 in Section 4 states that sometimes the existence of Nash equilibria can be established analytically near Nash steady states. Section 4 constructs Nash equilibria numerically, even when their existence cannot be established analytically.

3. Overview of steady states

To assess the stability properties of steady state equilibria, it is useful to begin by considering an economy with no fiat money ($M = 0$) and $\theta_i = \frac{1}{3}$. Iacopetta (2019) presents a more detailed analysis of such a simpler environment.

3.1. Economy with $M = 0$

Since the first two rows of $\mathbf{s}$ refer to the acceptance of fiat money – recall that the rows of $\mathbf{s}$ are associated to objects and the columns to types – we henceforth focus on the third row entries, $\mathbf{s}_3$, which show how agents i order good $i+1$ and good $i+2$. For instance, when $\mathbf{s}_3 = (0, 1, 0)$, type 2 trade $i+1$ for $i+2$ (i.e. 3 for 1), whereas types 1 and 3 do not. In addition, because $p_{1,m} = p_{2,m} = 0$, it is convenient to shorten $\mathbf{p}$ into $\hat{\mathbf{p}} = (p_{1,2}, p_{2,3}, p_{3,1})$.

Assume $c_1 < c_2 < c_3$ (model A of KW). There are eight possible combinations of (pure) strategies. Two of them are Nash equilibria:

$$\mathbf{s}_3 = (0, 1, 0) \text{ with } \hat{\mathbf{p}} = \frac{1}{3}\left(1,\ \frac{1}{2},\ 1\right), \tag{13}$$

if

$$\frac{c_3 - c_2}{u_1 \alpha} > p_{3,1} - p_{2,1} = \frac{1}{6}, \tag{14}$$

and

$$\mathbf{s}_3 = (1, 1, 0) \text{ with } \hat{\mathbf{p}} = \frac{1}{3}\left(\frac{1}{2}\sqrt{2},\ \sqrt{2} - 1,\ 1\right), \tag{15}$$

if

$$\frac{c_3 - c_2}{u_1 \alpha} < p_{3,1} - p_{2,1} = \frac{\sqrt{2}}{3}(\sqrt{2} - 1), \tag{16}$$

where the $p_{i,j}$ in (14) and (16) are evaluated in the respective steady states. These are usually referred as the *fundamental* and *speculative* steady states, respectively.

Rearranging the ranking of the storage cost to become $c_3 < c_2 < c_1$, one obtains steady states similar to those in Model B of KW. The equilibrium

$$\mathbf{s}_3 = (1, 0, 1) \text{ with } \mathbf{p} = \frac{1}{3}\left(\sqrt{2} - 1,\ 1,\ \frac{\sqrt{2}}{2}\right),$$

[8] In all our applications, $\Delta^i_{j,k}$ changed sign along the transition only in a finite set of instances t_l, $l = 1, \ldots, L$. Therefore $s^i_{j,k}$ is discontinuous at t_l. The value of $s^i_{j,k}(t_l)$ at such points of discontinuity is not relevant. Shevchenko and Wright (2004) make a similar observation.

[9] Because when $\Delta^i_{j,k} = 0$ the response $\sigma^i_{j,k}$ can take any value between 0 and 1, a more refined definition of the map $\mathcal{B}$ would be that this is a set of functions. Nevertheless, as we set $\sigma^i_{j,k} = 0.5$ when $\Delta^i_{j,k} = 0$ on a finite set, our definition is sufficient.

always exists. The equilibrium

$$\mathbf{s}_3=(0,1,1) \text{ with } \mathbf{p}=\frac{1}{3}\left(1,\ \frac{\sqrt{2}}{2},\ \sqrt{2}-1\right),$$

exists if

$$\frac{c_3-c_1}{u_2\alpha} > p_{3,2}\ -p_{1,2}=\frac{\sqrt{2}}{3}(1-\sqrt{2})\,, \tag{17}$$

and

$$\frac{c_2-c_3}{u_1\alpha} < p_{2,1}=\frac{\sqrt{2}}{6} \tag{18}$$

are satisfied, where $p_{1,2}$, $p_{3,2}$, and $p_{1,2}$ are evaluated on the $\mathbf{s}_3$=(0, 1, 1) steady state. Other equilibria emerge under other rearrangements of the storage costs. The next proposition states which steady state equilibria are globally stable.

Proposition 3. *With the possible exception of* $\mathbf{s}_3=(1,1,1)$*, under any other constant set of strategies* $\mathbf{s}_3=(s^1_{2,3},s^2_{3,1},s^3_{1,2})$*,* $\hat{\mathbf{p}}(t)$ *converges to a stationary distribution,* $\hat{\mathbf{p}}^*$*, from any* $\hat{\mathbf{p}}(0)$*.*[10]

3.2. Economy with M > 0

Consider now the full-fledged model with $M>0$. Adding fiat money into the model greatly increases the number of steady states that could qualify to be Nash equilibria. Considering that each type has six possible choices, there are 6^3 steady states to be verified. Fig. 1 illustrates how variations of M and δ_m affect the emergence of a particular equilibrium for a Model A economy ($c_1 < c_2 < c_3$). It considers equilibria with $\mathbf{s}_3=(0,1,0)$ or $\mathbf{s}_3=(1,1,0)$ — the same two types of equilibria reviewed above for the economy without fiat money. In reviewing the monetary equilibria, it is useful to keep in mind that money is accepted for liquidity reasons and to save on storage costs. Money holders, however, incur a seignorage tax. When this becomes sufficiently large, some may prefer to face longer waiting times for the preferred consumption good (lower liquidity), and to pay a higher storage cost, rather than holding fiat money. Since liquidity depends both on the distribution of commodities and on the magnitude of storage costs, it is conceivable that some do not accept money even when others do.

Fig. 1 shows that an $\mathbf{s}_3=(1,1,0)$ full monetary equilibrium, i.e., with $\mathbf{s}_1=\mathbf{s}_2=(1,1,1)$, emerges for a relatively low stock of fiat money and for low rates of seignorage. As seignorage gains in importance, however, fiat money becomes less desirable, to the point that a type 2 is no longer willing to sell good 1 against fiat money, that is, $s^2_{1,m}$ switches from 1 to 0, so that the set of strategies becomes $\mathbf{s}=\begin{pmatrix}1&1&1\\1&0&1\\1&1&0\end{pmatrix}$.

At higher levels of fiat money, good 3 loses its role of commodity money, even at low seignorage rates, because a type 1's odds of meeting type 3 holding good 1 shrink. Therefore, a type 1 no longer finds it convenient to pay the high storage cost of good 3. Hence, the Nash equilibrium is characterized by $\mathbf{s}=\begin{pmatrix}1&1&1\\1&1&1\\0&1&0\end{pmatrix}$.

At intermediate rates of seignorage, $s^2_{1,m}$ can be either 1 or 0 or both, that is, $\mathbf{s}_3=(1,1,0)$ and $\mathbf{s}_3=(0,1,1)$ — a case of multiple equilibria due to inflation.

4. Finding Nash equilibria

This section studies the conditions for obtaining a Nash equilibrium $(\mathbf{p}(t),\mathbf{s}(t))$ that converges to a steady state equilibrium, starting from an arbitrary initial distribution $\mathbf{p}(0)$. It begins with a proposition that deals with convergence in the neighborhood of a Nash steady state equilibrium.

Proposition 4. *Let* $(\mathbf{p}^*,\mathbf{s}^*)$ *be a Nash steady state equilibrium, with* $\mathbf{p}^*$ *being asymptotically stable for (1)–(3). There exists an* $\epsilon>0$ *such that, if* $\|\mathbf{p}_0-\mathbf{p}^*\|\le\epsilon$*, the pattern* $(\mathbf{p}(t),\mathbf{s}^*)$*, with* $\mathbf{p}(0)=\mathbf{p}_0$*, is a Nash Equilibrium.*[11]

Proof. See Appendix A ■

Clearly, when the initial condition $\mathbf{p}(0)$ is outside of the small range of the Nash steady state, $(\mathbf{p}^*,\mathbf{s}^*)$, the Nash set of strategies $\mathbf{s}(t)$ may be different than $\mathbf{s}^*$. In such a case, to evaluate whether an $\mathbf{s}(t)$ is a Nash solution, one needs to verify whether any agent has an incentive to deviate from such an $\mathbf{s}(t)$. In terms of (10), one needs to check if there is any gap between $\mathbf{s}^i(t)$ and $\boldsymbol{\sigma}^i(t)$. The best response $\boldsymbol{\sigma}^i(t)$ can be computed on the basis of the value functions $V_{i,j}(t)$ defined in (10). Eq. (7) specifies the time variation of $V_{i,j}(t)$. Unfortunately, the initial value $V_{i,j}(0)$ is unknown. What is known, however, is the value of $V_{i,j}$ on the Nash steady state $(\mathbf{p}^*,\mathbf{s}^*)$. This information can be used to integrate (7) backward in time, starting from a neighborhood of the Nash steady state. We then proceed as follows.

First, we integrate the system (1)–(3) starting from the initial condition $\mathbf{p}(0)$, under the set of strategies $\mathbf{s}(t)$, an operation that yields $\mathbf{p}(t)$, for $0\le t\le T$, where T must be sufficiently large so that $\|\mathbf{p}(T)-\mathbf{p}^*\|\le\epsilon$. Let T_ϵ be a T that satisfies this constraint.

Second, we set $\mathbf{V}_i(T_\epsilon)=\mathbf{V}^*_i$, where $\mathbf{V}_i(t)=\{V_{i,j}(t)\}_{j=i+1,i+2,m}$, and use (7) to compute $\mathbf{V}_i(t)$ for $t<T_\epsilon$. In other words, we use $\mathbf{V}^*_i$ as final condition and integrate (7) backward in time. The resulting $\mathbf{V}_i(t)$, however, is not yet necessarily the true value function because T_ϵ is finite.

Additional details must still be verified to arrive to the true value function. Let $\mathbf{V}_i(t,T,\mathbf{W})$ be the unique solution of (7), for a given $(\mathbf{s}(t),\mathbf{p}(t))$, with $t\le T$ under the condition that for $t=T$, $\mathbf{V}_i(T)=\mathbf{W}$, i.e., $\mathbf{V}_i(T,T,\mathbf{W})=\mathbf{W}$. The proposition below states that it is reasonable to pick a $\mathbf{W}=\mathbf{V}^*_i$ as an initial condition to integrate the system (7) backward in time.

Proposition 5. *Consider a pattern* $(\mathbf{s}(t),\mathbf{p}(t))$ *that converges to a steady state Nash equilibrium* $(\mathbf{s}^*,\mathbf{p}^*)$*. Let* $\mathbf{V}^*_i$ *be the value function evaluated at the Nash steady state* $(\mathbf{s}^*,\mathbf{p}^*)$*, and let* $\mathbf{V}_i(t,T,\mathbf{W})$ *be the unique solution of (7) with* $\mathbf{V}_i(T,T,\mathbf{W})=\mathbf{W}$*. For every* $t\le T$

$$\|\mathbf{V}_i(t)-\mathbf{V}_i(t,T,\mathbf{V}^*_i)\|\le\sqrt{3}e^{-\rho(T-t)}\|\mathbf{V}_i(T)-\mathbf{V}^*_i\|\,.$$

Proof. See Appendix A. ■

The fact that $\|\mathbf{p}(T_\epsilon)-\mathbf{p}^*\|\le\epsilon$ implies that $\|\mathbf{V}_i(T_\epsilon)-\mathbf{V}^*_i\|=O(\epsilon)$. Proposition 5 states that choosing $\mathbf{V}^*_i$ as a boundary value, (7) yields a good approximation $\mathbf{V}_i(t,T_\epsilon,\mathbf{V}^*_i)$ for $\mathbf{V}_i(t)$. The response $\boldsymbol{\sigma}^{i,\epsilon}(t)$ derived from $\mathbf{V}_i(t,T_\epsilon,V^*_i)$ is then an approximation of the best response $\boldsymbol{\sigma}^i$. In particular, $\boldsymbol{\sigma}^{i,\epsilon}$ is a piecewise constant function with a finite number of switching times t^ϵ_h, $h=1,\dots,H$, where one of the $\sigma^{i,\epsilon}_{j,k}$ changes from 0 to 1 or from 1 to 0. As ϵ approaches zero, $\boldsymbol{\sigma}^{i,\epsilon}$ converges to $\boldsymbol{\sigma}^i$ in the sense that $\lim_{\epsilon\to0}t^\epsilon_h=t_h$, where t_h, for $h=1,\dots,H$, is the finite set of witching times of $\boldsymbol{\sigma}^i$. In short, we have

$$\boldsymbol{\sigma}^i=\mathcal{B}(\mathbf{s})=\lim_{\epsilon\to0}\boldsymbol{\sigma}^{i,\epsilon}. \tag{19}$$

We deal with the problem of finding a fixed point for the map $\mathcal{B}$ (see (12)) by designing a simple iterative scheme. It is convenient to define $\mathbf{V}(t)=(\mathbf{V}_1(t),\mathbf{V}_2(t),\mathbf{V}_3(t))$. The iteration

[10] The proof can be found at the following link: http://people.math.gatech.edu/\char126\relaxbonetto/Deposit/App.pdf.

[11] We define $\|\mathbf{p}\|=\sqrt{\sum_{i,j}p^2_{i,j}}$.

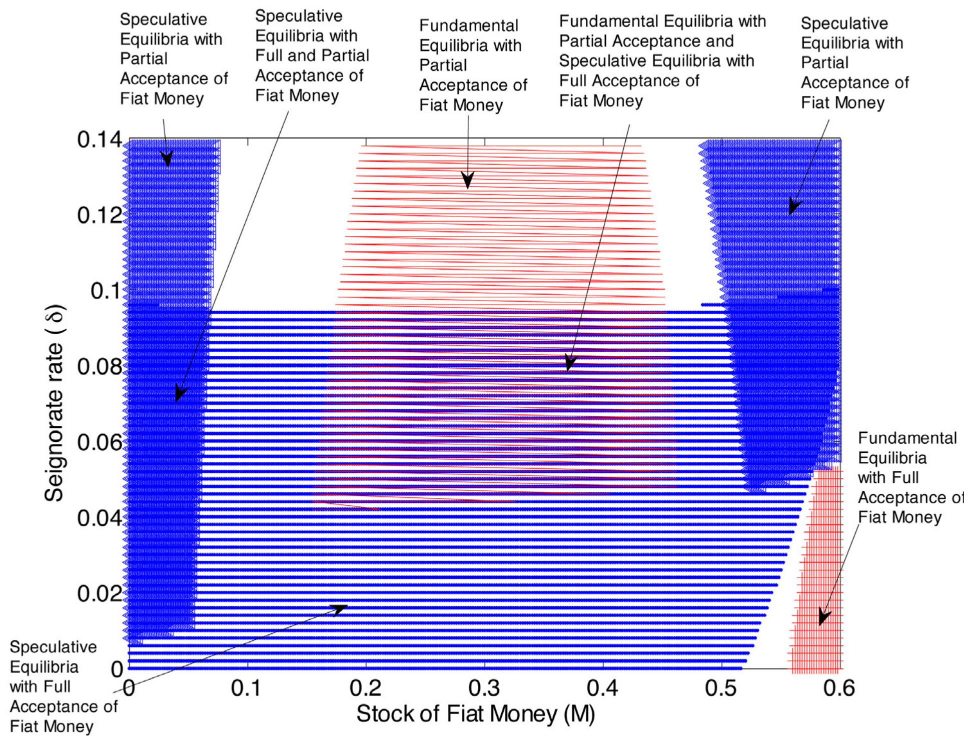

Fig. 1. Overview of equilibria. Note. There are four types of equilibria: Speculative equilibria $s_3 = (1,1,0)$ with full acceptance of fiat money are in the dark dotted region; Fundamental equilibria $s_3 = (0, 1, 0)$ with full acceptance of fiat money are in the + light color region on the lower-right side; Speculative equilibria with partial acceptance of fiat money ($s^2_{1,m} = 0$ and $s^i_{j,m} = 1$ when $i \neq 2$ and $j \neq 1$) are represented by the signs $<$; and Fundamental equilibria with partial acceptance of money are represented by the sign –. In some regions two equilibria coexist. The population is equally divided between the three types of agents ($\theta_i = \frac{1}{3}$). The remaining parameter values are in Table 1, Model A.

starts with a guess, $\mathbf{s}_0(t)$, and then determines the best response $\boldsymbol{\sigma}_0(t)$ to $\mathbf{s}_0(t)$, where $\boldsymbol{\sigma}_0 = \mathcal{B}(\mathbf{s}_0)$ and the associated pattern $\mathbf{p}_0(t)$. If $\boldsymbol{\sigma}_0(t) = \mathbf{s}_0(t)$, the iteration stops and $(\mathbf{s}_0(t), \mathbf{p}_0(t))$ is the Nash equilibrium. Otherwise, the iteration continues: It sets a new guess $\mathbf{s}_1(t) = \boldsymbol{\sigma}_0(t)$ and calculates a new path $(\mathbf{s}_1(t), \mathbf{p}_1(t))$. In general, the iteration generates a sequence $\boldsymbol{\sigma}_n = \mathcal{B}(\mathbf{s}_n)$. If the sequence $\mathbf{s}_n(t)$ converges to a $\mathbf{s}(t)$ then the couple $(\mathbf{s}(t), \mathbf{p}(t))$ is a Nash equilibrium.

More specifically, to find a Nash equilibrium that starts from $\mathbf{p}(0)$ and converges to a Nash steady state $(\mathbf{s}^*, \mathbf{p}^*)$ we:

1. determine the Nash steady state $(\mathbf{s}^*, \mathbf{p}^*)$;
2. choose an initial guess $\mathbf{s}_0(t)$ for $\mathbf{s}(t)$ (for instance, $\mathbf{s}_0(t) = \mathbf{s}^*$);
3. integrate (1)–(3) forward in time, with $\mathbf{s}(t) = \mathbf{s}_0(t)$, until the solution $\mathbf{p}_0(t)$ is sufficiently close to the steady state $\mathbf{p}^*$;
4. compute $\mathbf{V}(t)$, by integrating (7) backward in time (with $\mathbf{V}^*$ as final condition) and contemporaneously determining $\boldsymbol{\sigma}_0(t)$ through (11), assuming $\mathbf{p}(t) = \mathbf{p}_0(t)$ and $\mathbf{s}(t) = \mathbf{s}_0(t)$;
5. set $\mathbf{s}_1(t) = \boldsymbol{\sigma}_0(t)$ as the new guess for $\mathbf{s}(t)$ and compute a new pattern $(\mathbf{s}_1(t), \mathbf{p}_1(t))$ and a new best response $\boldsymbol{\sigma}_1(t)$;
6. repeat steps 3 through 5 to generate a sequence $(\boldsymbol{\sigma}_n(t), \mathbf{s}_n(t), \mathbf{p}_n(t))$;
7. stop the procedure when the difference between $\boldsymbol{\sigma}_n(t)$ and $\mathbf{s}_n(t)$ is smaller than a predetermined error ϵ.[12]

Two observations are in order. First, there are no issues of instability when computing $\mathbf{p}(t)$ and $\mathbf{V}(t)$: The system (1)–(3) is stable when integrated forward in time and so is (7) if integrated backward in time. The iteration does not need to converge, but if it does, it necessarily converges to a fixed point of $\mathcal{B}$. In the case of non-convergence, more refined iterative schemes or a Newton–Raphson method could be employed. Nevertheless, in all our numerical experiments (Section 5) this simple iterative algorithm delivered a fixed point. Second, the design of the algorithm presents similarities to the stable manifold theorem for ordinary differential equations (see Appendix B for a formal discussion). It differs, however, from the standard approaches employed to study transitional dynamics of macroeconomic and growth models. Indeed, it is common to compute an equilibrium pattern by integrating a system of differential equations that describe the equilibrium conditions of the economy backward in time, starting from a neighborhood of the steady state (see Brunner and Strulik, 2002). This approach usually works well when the dimension of the system is small. At high dimensions it is sometimes possible to approximate the manifold in the neighborhood of the steady state by means of projection methods (see McGrattan (1999), and Mulligan and Sala-i-Martin (1993)). Nevertheless, when the dimension of the system is large, constructing the manifold in regions away from the steady state is generally very problematic — a serious limitation when the choice of an initial condition away from the steady state is an important aspect of the exercise.

5. Numerical experiments

This section proposes a few applications for the dynamic analysis. First, it illustrates the transition from partial to full acceptance of fiat money. Second, it studies the effects of changes in the rate of seignorage. It then discusses the issues of multiple equilibria related to seignorage and to the distribution of the population across the three types. Finally, it briefly reviews equilibria in Model B.

[12] We define the distance between $\boldsymbol{\sigma}_n$ and $\mathbf{s}_n$ as the $\max_{i=1,\ldots,K} |t_i - \tau_i|$, where t_i is the switching time in $\boldsymbol{\sigma}_n$ and τ_i is that in $\mathbf{s}_n$. The $(\mathbf{s}(t), \mathbf{p}(t))$ obtained by taking $\epsilon \to 0$ is a Nash equilibrium.

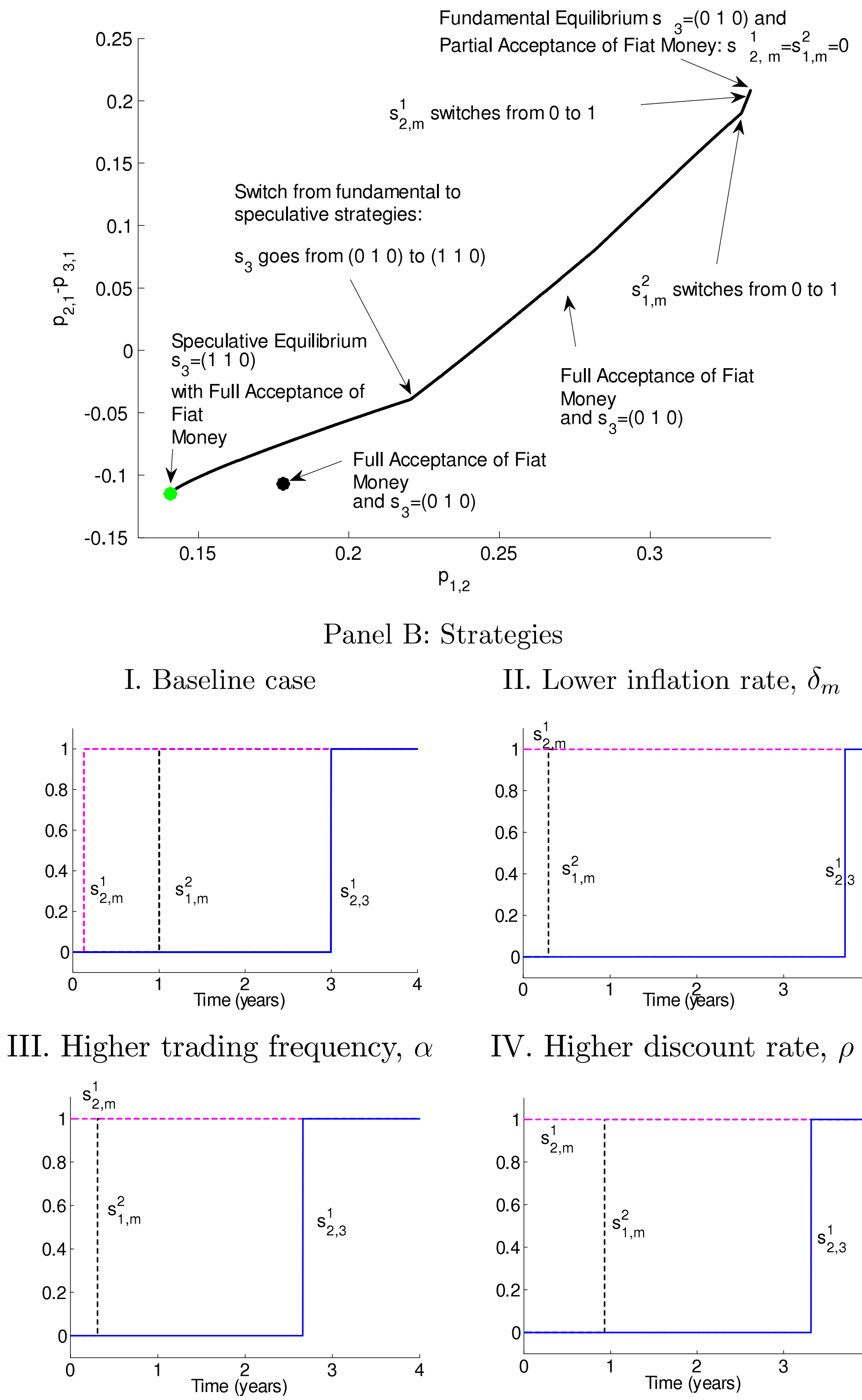

Fig. 2. Acceptance of commodity and Fiat Money. Note: Panel A shows the convergence to a $s_3 = (1, 1, 0)$ steady state equilibrium with full acceptance of money. The initial condition $p(0) = (\theta_1, 0, 0, 0, \frac{M}{4})$, and $\theta_i = \frac{1}{3}$. Panels B.I–B.IV show the possible switch of $s^1_{2,m}$, $s^2_{1,m}$ and of $s^1_{2,3}$ from 0 to 1, in the baseline case (B.I), and then for the cases $\delta_m = 0.06$, $\alpha = 1.2$, $\rho = 0.06$, in panels B.II, B.III, and B.IV, respectively.

5.1. Partial and full acceptance of Fiat Money

Because the conditions for the full acceptance of money in a steady state are different than those in other regions of the inventory space, an economy may go through a phase, while converging to a full monetary equilibrium, in which some do not accept fiat money. For a start, over the transition, the degree of acceptability of a low-storage commodity may simply decline, and thus favor the acceptability of fiat money. In addition, the cost of seignorage, given by $V_{i,m} - V_{i,i+1} - D_i$, may also decline over the transition — the production cost, D_i, is constant over time but the difference $V_{i,m} - V_{i,i+1}$ is not. It could be, for instance, that during the transition, the liquidity of commodity $i+1$ drops, implying a reduction in the cost of seignorage. Panel A of Fig. 2 illustrates such a scenario in the phase diagram. Initially, type 1 prefers good 2 to fiat money s, and type 2 prefers good 1 to fiat money, and only good 1 plays the role of commodity money. The economy eventually converges to a full monetary equilibrium and, furthermore, good 3 acquires the role of money. Throughout the transition, not only fiat money is accepted by a larger share

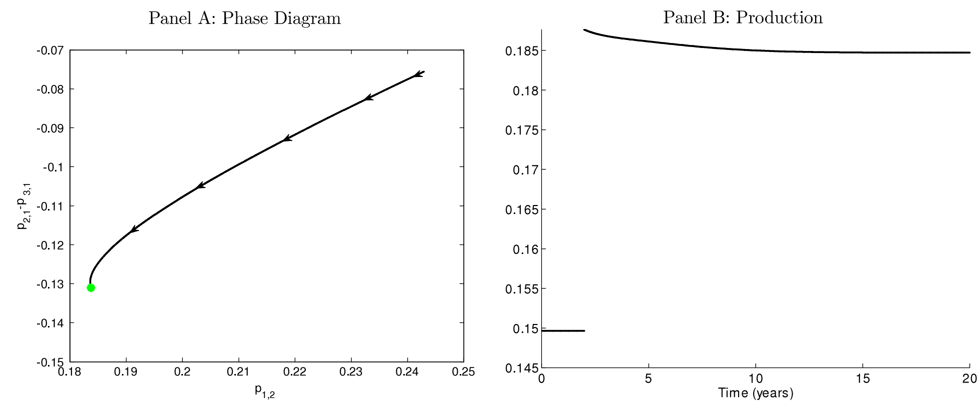

Fig. 3. Reduction of seignorage. Note: The seignorage rate, δ_m, goes from 0.1 to 0.02. The economy transits from a steady state equilibrium in which $\mathbf{s}_1 = (1, 1, 1)$, $\mathbf{s}_2 = (0, 1, 1)$, and $\mathbf{s}_3 = (0, 1, 0)$, to new equilibrium in which $\mathbf{s}_1 = \mathbf{s}_2 = (1, 1, 1)$, and $\mathbf{s}_3 = (1, 1, 0)$. The stock of fiat money is $M = 0.3$ and $\theta_i = \frac{1}{3}$. The remaining parameters are in Table 1, Model A.

of the population, but good 3 acquires the role of commodity money, as type 1 agents switch from fundamental to speculative strategies. The emergence of commodity money is driven by the change in liquidity of good 3 relative to good 2: Fig. 2 shows that along the transition path, $p_{2,1}-p_{3,1}$ declines, implying that agents 1 tend to shift their demand from good 2 to good 3. Next, with the aid of Fig. 2B, we study how the switching time is affected by the rate of inflation, the frequency of trade, and by the individual's discount rate.

Inflation (Fig. 2B.II). Lower rates of seignorage hasten the emergence of money. The lower risk of confiscation induces some individuals of type 1 and of type 2 to abandon their strategies of holding inventories earlier, accepting to trade against fiat money when the occasion arises. By doing so, they shorten the waiting time for acquiring their consumption goods under the state in which fiat money is not confiscated, but, at the same time, they give up the option of being insured against the risk of inflation sooner.

Higher trading frequency (Fig. 2B.III). A higher matching rate facilitates trade both when individuals have fiat money and when they engage in indirect trading. Nevertheless, fiat money is relatively more accepted than other commodities; this is the object that leads relatively more quickly to the consumption good. Therefore, for a given inflation rate, the trading frequency favors the emergence of fiat money over the transition.

The discount rate (Fig. 2B.IV). The effects of a higher rate of discount are somewhat similar to those associated to a higher trading frequency. This observation has been used, for instance, in Araujo and Guimaraes (2014) to proxy the acceleration of the frequency of trade with an increase in the individuals' discount rate. In the current set framework, however, there are some interesting differences. While in the high-trade-frequency economy the switches to both fiat and commodity money occur earlier relative to the baseline economy, in the high-discount-rate economy only the switch to fiat money takes place earlier. Conversely, in the high-discount economy there is a delay in the adoption of commodity 3 as money relative to the baseline economy. The results differ in the two comparative dynamics because the cost of storage hits agents immediately, while that of inflation is projected in the future. In a high-discount economy, on the one hand, the storage cost becomes more relevant in relation to the future advantage of obtaining a highly liquid object. For this reason, the adoption of commodity money 3 is postponed. On the other hand, individuals put a lower weight on the risk that their fiat money will be confiscated. Therefore, the liquidity value of fiat money becomes more important than the risk of inflation and it is accepted sooner.

5.2. A monetary reform

This section studies the role of inflation in more detail. It is a long-standing tenet of economics that inflation can create inefficiencies because it distorts the choices of individuals. To understand how people's behavior is affected by seignorage, consider an economy that is currently on a full monetary steady state equilibrium, and where only good 1 is also used as commodity money. Through a reduction of the seignorage rate, the government may pull the economy out of this equilibrium and send it into a full monetary equilibrium in which good 3 is also accepted as money. In the latter equilibrium, on average, agents trade more frequently and produce at a faster rate than in the former one. For instance, a 2 percentage reduction of δ_m, from the initial state $M = 0.3$ and $\delta_m = 0.1$, would be sufficient to accomplish the task (see Fig. 1). The dynamic consequences of the shock are depicted in the phase diagram of Fig. 3. Because the drop in the seignorage rate increases the value of fiat money relative to that of all other commodities, type 2 agents are induced to sell good 1 against fiat money. In addition, because the gap $p_{2,1}$–$p_{3,1}$ declines over the adjustment period to the new equilibrium, type 1 agents, in anticipation of such liquidity change, immediately switch from fundamental to speculative strategies ($\mathbf{s}_3$ turns from $(0, 1, 0)$ to $(1, 1, 0)$). As a result, production booms right after the shock and then stabilizes at a higher level relative to that of the initial equilibrium (see Fig. 3b).

5.3. Inflation and beliefs

It is often argued that the effects of inflation on production depend on the coordination of beliefs. This conjecture, in our framework, emerges in Fig. 1, which reviews the type of equilibria associated with different combinations of M and δ_m. This figure shows that in some regions of the (M, δ_m) space, two steady state equilibria exist. For instance, when $M = 0.3$, and δ_m is between 5 and 9 percent, the full monetary speculative equilibrium coexists with a fundamental equilibrium in which $s^2_{2,m} = 0$. For an inflation rate above 9 percent, however, there are unique fundamental equilibria with partial acceptability of money. Hence, in an economy that is initially in such a state (with $s^2_{2,m} = 0$), a reduction of the seignorage rate, from 10 to 6 percent, for example, may or may not induce type 1 agents to switch from fundamental to speculative strategies ($\mathbf{s}_3$ may or may not change from $(0, 1, 0)$ to $(1, 1, 0)$). Agents may keep their actions coordinated on the current fundamental equilibrium, in which case the intervention policy generates only marginal changes in the economy. Conversely, they may coordinate their actions on

Table 1
Baseline parameters.

Model	Discount	Matching	Utility		Storage costs		
	δ	α	u_i	D_i	c_1	c_2	c_3
A	0.03	1	1	0.028	0.03	0.1	0.2
B	0.03	1	1	0.028	0.1	0.05	0.03

the speculative equilibrium, in which case the adjustment process would be very similar to that depicted in Fig. 3.

In brief, a modest reduction of the seignorage rate associated with a dose of optimism can be effective in stirring up production. But if agents are unresponsive to relatively modest changes in the seignorage rate, the government would need to implement a more radical monetary reform, and be prepared to give up a larger share of its current seignorage revenue — in our example, below $\delta_m = 0.05$ there is a unique equilibrium with $\mathbf{s}_3 = (1, 1, 0)$.

5.4. Uneven distribution of types and multiple equilibria

The distribution of the population across types affects the emergence of a particular equilibrium. Wright (1995) established the existence of multiple equilibria in a Model A economy without fiat money. In particular, when the share of type 3 agents is relatively high, the speculative equilibrium $\mathbf{s}_3 = (1, 1, 0)$ coexists with one in which all three agents flip their strategies, that is, a $\mathbf{s}_3 = (0, 0, 1)$ equilibrium. Seignorage promotes the emergence of additional equilibria. Fig. 4a shows how the values of θ_i condition the emergence of a particular equilibrium for a given stock of real balances, and of the seignorage rate. With a symmetric distribution $\theta_i = \frac{1}{3}$, under the current specification ($M = 0.3$, $\delta_m = 0.02$; see also Table 1), there exists a unique full monetary speculative equilibrium. A unique full monetary fundamental equilibrium is observed when type 3 individuals become less numerous and are replaced by type 1 individuals. If instead type 1 replaces type 2 agents, multiple equilibria with partial and full acceptability of money appear. Unique equilibria with partial acceptability of fiat money are observed when θ_1 is high and θ_3 is low: a situation in which type 2 easily trades 1 for 2.

At higher seignorage rates, regions of partial acceptability of fiat money expand. For instance, a comparison of Fig. 4a and b reveals that when the seignorage rate goes from 2 to 10 per cent, the full monetary fundamental equilibrium disappears, and an overlap between full and partial monetary speculative equilibria is more commonly observed.

Multiple Equilibria. The presence of multiple monetary steady states does not necessarily imply the existence of multiple Nash equilibria. In principle, it could be that once the initial condition is specified, the pattern converges to one and only one steady state. Fig. 5 clarifies, however, that in our environment there is multiplicity: two economies with the same set of parameters and the same initial condition coordinate on different steady-state equilibria.

5.5. Uneven distribution of types and dynamics in Model B

This section briefly discusses the acceptability of fiat money in a Model B economy, for different values of θ_i. The effects of a rise in the seignorage rate can be learned by comparing the equilibria in Fig. 6b where $\delta_m = 0.1$ and those in Fig. 6a where $\delta_m = 0.02$. In the low-inflation economy, larger regions of the full monetary equilibria overlap with similar equilibria in which $s^1_{3,m} = 0$. In the space just below the 45-degree line, there are only full monetary fundamental equilibria $\mathbf{s}_3 = (1, 0, 1)$ because the scarcity of type 1 agents reduces the liquidity value of good 3 — there are too few middle-men that bring good 3 from type 2 to type 3 agents. Indeed, good 3 is dominated by fiat money even with a seignorage rate three times larger than c_3. For a more balanced distribution of the population, however, at sufficiently high rates of seignorage, equilibria with partial acceptability of money ($s^1_{3,m} = 0$) become unique.

As with model A, the liquidity conditions and the cost of seignorage change over time, implying that, for a given specification of the economy, the fraction of the population that accepts money changes over the transition to the Nash steady state equilibrium. Fig. 7 shows a particular scenario where, as the economy converges to the $\mathbf{s}_3 = (1, 0, 1)$ full monetary steady state equilibrium, type 3 agents switch their strategies with respect to fiat money when holding good 2. As the transition progresses, both good 2 and fiat money become more valuable (see middle plot of Fig. 7) as their liquidity improves, but the value of fiat money increases more rapidly and eventually catches up with the value good 2.

6. Welfare

One standard question of monetary economics is whether the acceptance of fiat money improves the allocation of resources and stimulates production. The presence of matching frictions and the assumption that agents incur a cost in holding commodities gives fiat money a potential positive role. Nevertheless, it also comes with costs at the levels of both the individual and society. At the individual level, the risk of confiscation looms. At the society level, fiat money reduces the availability of commodities — money chases away consumption goods. To explore the welfare implications of introducing fiat money and of altering seignorage rates, we use, as KW, a utilitarian welfare criterion: Given the highly symmetric type of environment it is unlikely to yield a Pareto improvement from any given state. The payoff for a type i agent is calculated as a weighted average of $V_{i,j}$:

$$W_i(t) = \frac{1}{\theta_i}(p_{i,i+1}V_{i,i+1} + p_{i,i+2}V_{i,i+2} + p_{i,m}V_{i,m}).$$

Therefore, the welfare of the whole society is simply the average of the three groups' payoffs:

$$W(t) = \sum_i \theta_i W_i(t).$$

As mentioned in the introduction, the government derives utility from consuming goods. These are purchased through the seignorage tax $\delta_m M$. One may argue that the government also cares about the welfare of the population. This could reflect a genuine interest in the society's well-being, or more simply the desire to maintain the population's electoral support. The government welfare function is then

$$W^G(t) = (1 - \lambda)Q(t) + \lambda W(t),$$

where

$$Q(t) = M \int_t^{+\infty} e^{-\rho^G(\tau - t)} \delta_m(\tau) d\tau,$$

and where λ weighs the society's welfare within the government's objective function. For the sake of illustration, we consider an altruistic government ($\lambda = 1$) that wants to know how the level of M or the rate of seignorage affects the population.

Real Balances. Fig. 8a shows the steady-state welfare levels for different levels of fiat money and zero seignorage. When the stock of fiat money is relatively large, any further increase tends to make people, on average, worse-off because the chase-away-good effect largely dominates. Observe that a change in M can induce W_1, W_2 and W_3 to move in different directions, indicating

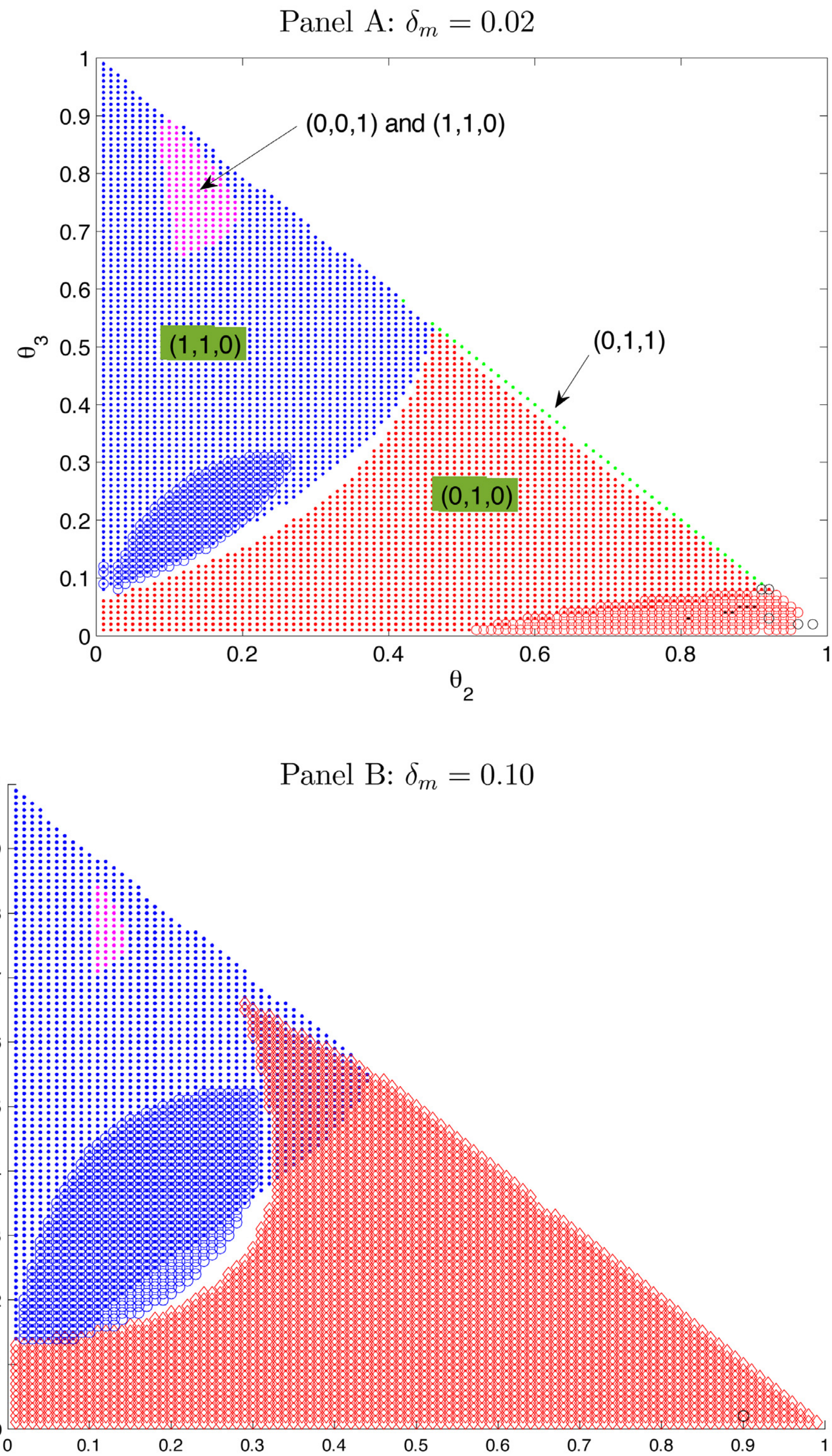

Fig. 4. Overview of the steady state equilibria, Model A. Note: On equilibria represented by a dot, $s^i_{j,m} = 1$ and: $\mathbf{s}_3 = (0, 1, 0)$ (red dot); $\mathbf{s}_3 = (1, 1, 0)$ (blue dot); $\mathbf{s}_3 = (0, 0, 1)$ (magenta dot); $\mathbf{s}_3 = (0, 1, 1)$ (green dot). On equilibria represented by a circle, $s^2_{1,m} = 0$ and: $\mathbf{s}_3 = (0, 1, 0)$ (red circle); $\mathbf{s}_3 = (1, 1, 0)$ (blue circle), and $\mathbf{s}_3 = (1, 1, 0)$ (black circle). The triplets in parentheses in plot A denote $\mathbf{s}_3$. M = 0.3 in both plots. For the remaining parameter values see Table 1, Model A. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

a conflict of interest in the society about which real balance level is the most desirable. Type 3 individuals, for instance, prefer a lower level of fiat money than the other two groups. As they carry the low storage cost good, they are more preoccupied, relative to the other two groups, by the displacement of commodities caused by a further increase in fiat money than they are pleased by the savings in storage costs.

Seigniorage. When the government confiscates fiat money, it clearly reduces the welfare of the targeted individuals by $V_{i,m} - V_{i,i+1} - D_i$. But it also alters the odds that an individual is

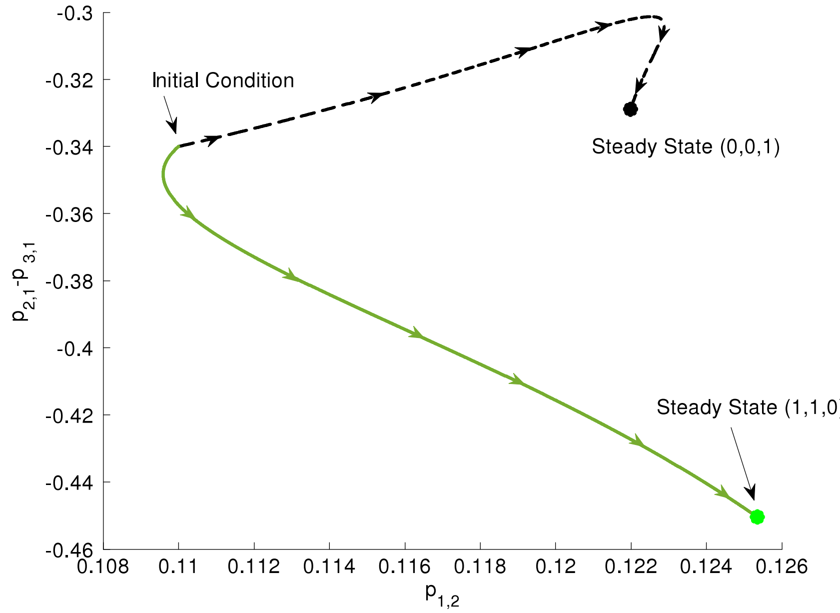

Fig. 5. Multiple equilibria. Note: The distribution of the population is as follows: $\theta_1 = 0.24$, $\theta_2 = 0.16$, and $\theta_3 = 0.7$. The rate of seignorage and the stock of fiat money is 0.05 and 0.2, respectively. The storage costs are: $c_1 = 0.03$, $c_2 = 0.1$, and $c_3 = 0.2$. The initial condition is $\mathbf{p}(0) = (0.11, 0.06, 0.38, 0.02, 0.06)$.

able to exchange his commodity against money. While a private agent in a match may refuse to buy, when a government agent carrying money meets a private agent, there is always an exchange. Hence, seignorage can act as a stimulus for production. However, seignorage also generates some income redistribution. First, as noted earlier, the cost of seignorage differs across types: Because a type 3 agent who relinquishes fiat money produces the good with the lowest storage cost, his burden of seignorage is lighter than that of the other two types. Second, because real balances are not generally held in equal proportions across the three groups, the probability of being hit by seignorage differs across types. Similarly, the probability that the government will purchase a commodity depends also on how commodities are distributed across types. Fig. 8b, for instance, shows that over a certain range of seignorage, an increase in seignorage causes a decline in W_1 and W_2 and an increase in W_3. Interestingly, the figure also shows that an increase in seignorage can boost everybody's welfare W_i when starting from low levels of seignorage.

To make sense of these results, it is important to point out the two effects of inflation in this framework. The first is purely redistributive. In a given equilibrium balances are unevenly distributed across the three types of individuals. Hence a flat tax on balances is bound to strike the three groups unevenly. Similarly, the government purchases goods at a higher frequency from types that hold relatively more of them. A second effect is the acceleration of the frequency of production and trade. Although the government consumes the confiscated goods, the acceleration of trade can be large enough to more than compensate, at an aggregate level, for the cost of seignorage. Specifically, if the government purchases goods from individuals who value fiat money more than individuals from whom the money had been confiscated, and if such a difference is large enough to compensate for the production cost, then the seignorage can increase global welfare. The experiment in Fig. 8b suggests that such a possibility may arise only at low levels of seignorage where the gap between the value of money and the commodity value is more likely to be large.[13] While the redistribution and the acceleration of production effects are well aligned with other monetary models, the positive effect of inflation, albeit in a limited inflation range, is more specific to the current framework.

7. Further research

The set up of the problem (Section 2) and the procedure to find Nash equilibria (Section 4) are valid for a more general search model with N goods, and N types of agents, as described, for instance, in Aiyagari and Wallace (1991, 1992, see Appendix A). The analysis could also be adapted to allow for multiple holdings (Molico, 2006; Lagos and Rocheteau, 2009; Chiu and Molico, 2010) and to study the dynamics of indivisible-asset models in which heterogeneity is an essential ingredient, such as studies of the middlemen by Rubinstein and Wolinsky (1987), international currency by Matsuyama et al. (1993), banking by Cavalcanti and Wallace (1999), and over-the-counter financial markets by Duffie et al. (2005). Future research could also clarify whether the positive welfare effects of inflation at low ranges still stand when an intensive margin of trade is added to the model.

Appendix A. Proofs and derivations

This Appendix contains the proofs of Propositions 1 to 5. While the statements of Propositions 1, 2, 4, and 5 refer to a KW economy with three types of commodities, fiat money, and three types of agents, the proofs we present here hold for a more general environment with n objects and n types of agents. Specifically, the size of the matrix $\mathcal{A}^i$, defined in (25) of this Appendix, can be augmented to consider more objects, and the index i, associated with types of individuals, can run up to an n larger than three.

Proof of Proposition 1. Differentiation of (6) with respect to t yields

$$\dot{V}_{i,j}(t) = -v_{i,j}(\mathbf{p}(t)) + \rho V_{i,j}(t) + \int_t^\infty e^{-\rho(\tau-t)} \sum_l \dot{\pi}^i_{l,j}(\tau,t) v_{i,l} \times (\mathbf{p}(\tau)) d\tau, \tag{20}$$

[13] Integrating the utility the government derives from seignorage, would enlarge the range within which inflation generates positive welfare effects.

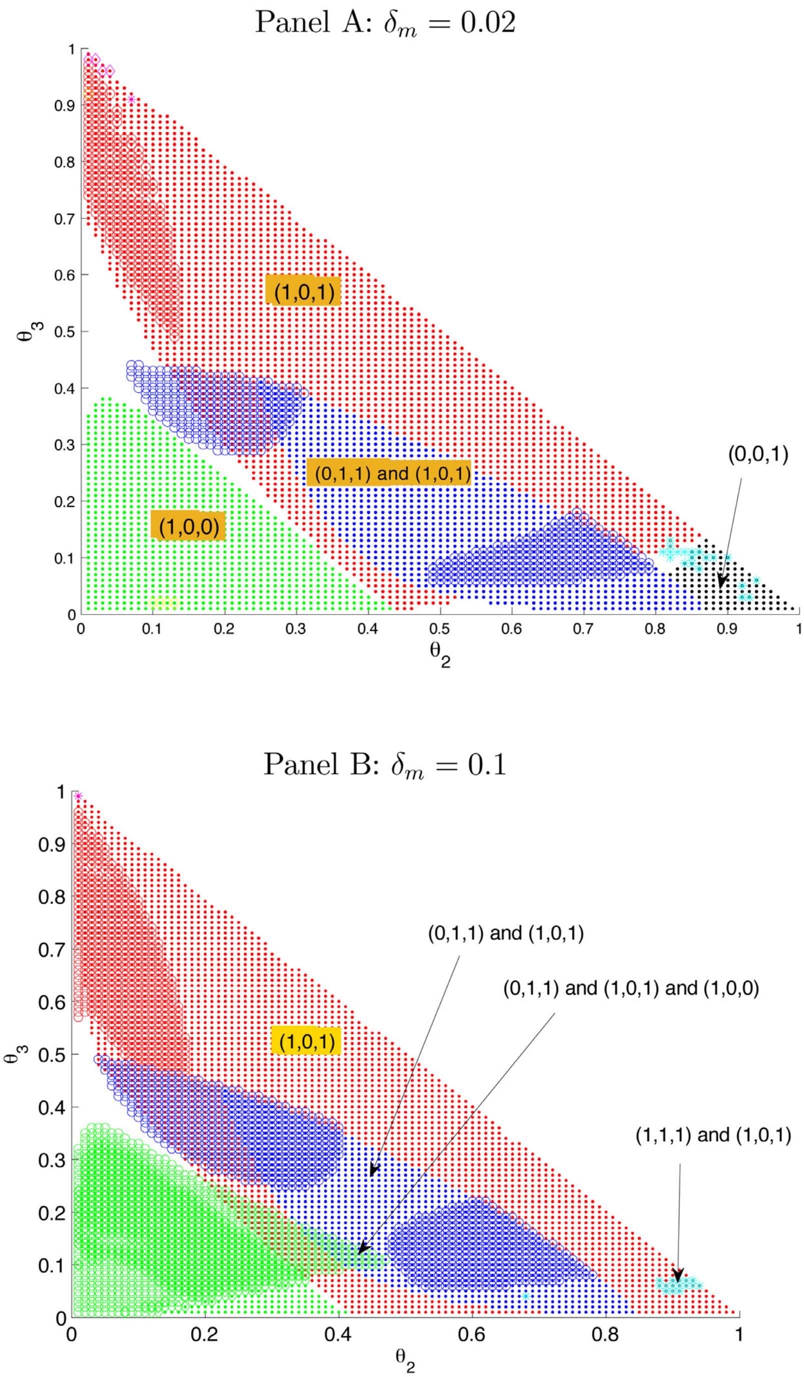

Fig. 6. Overview steady state equilibria, Model B. Note: On equilibria represented by a dot, $s^i_{j,m} = 1$ and: $\mathbf{s}_3 = (1, 0, 1)$ (red dot); $\mathbf{s}_3 = (0, 1, 0)$ (blue dot); $\mathbf{s}_3 = (1, 0, 0)$ (green dot); and $\mathbf{s}_3 = (0, 0, 1)$ (black dot). On equilibria represented by a circle, $s^2_{1,m} = 0$ and: $\mathbf{s}_3 = (1, 0, 1)$ (red circle); $\mathbf{s}_3 = (0, 1, 0)$ (blue circle), $\mathbf{s}_3 = (1, 0, 0)$ (green circle); and $\mathbf{s}_3 = (0, 0, 1)$ (black circle). The triplets in parentheses in the two plots denote $\mathbf{s}_3$. M = 0.3 on both plots. For other parameters values see Table 1, Model B. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where

$$v_{i,j}(\mathbf{p}) = \sum_{i'} p_{i',i} s^{i'}_{j,i} u_i - c_j \tag{21}$$

(when $j = m$, c_m stands for $\delta_m D$) is the expected utility from consumption, net of storage cost, for an agent of type i with good j, and where $\dot{\pi}^i_{k,j}(\tau, t) \equiv \frac{d}{dt}\pi^i_{k,j}(\tau, t)$. To derive $\frac{d}{dt}\pi^i_{k,j}(\tau, t)$, first observe that expressions similar to (1)–(3) imply that

$$\begin{aligned} \frac{d}{d\tau}\pi^i_{i+1,j}(\tau, t) =& \alpha \Bigg\{ \sum_{i'} \sum_{k} \pi^i_{k,j} p_{i',i+1} \sigma^i_{k,i+1} s^{i'}_{i+1,k} + \sum_{i'} \pi^i_{k,j} p_{i',i} s^{i'}_{i,k} \\ & - \sum_{i'} \sum_{k} \pi^i_{i+1,j} p_{i',k} \sigma^i_{i+1,k} s^{i'}_{k,i+1} \Bigg\} \\ & - \delta_g \pi^i_{i+1,j} + \delta_m \pi^i_{m,j} \end{aligned} \tag{22}$$

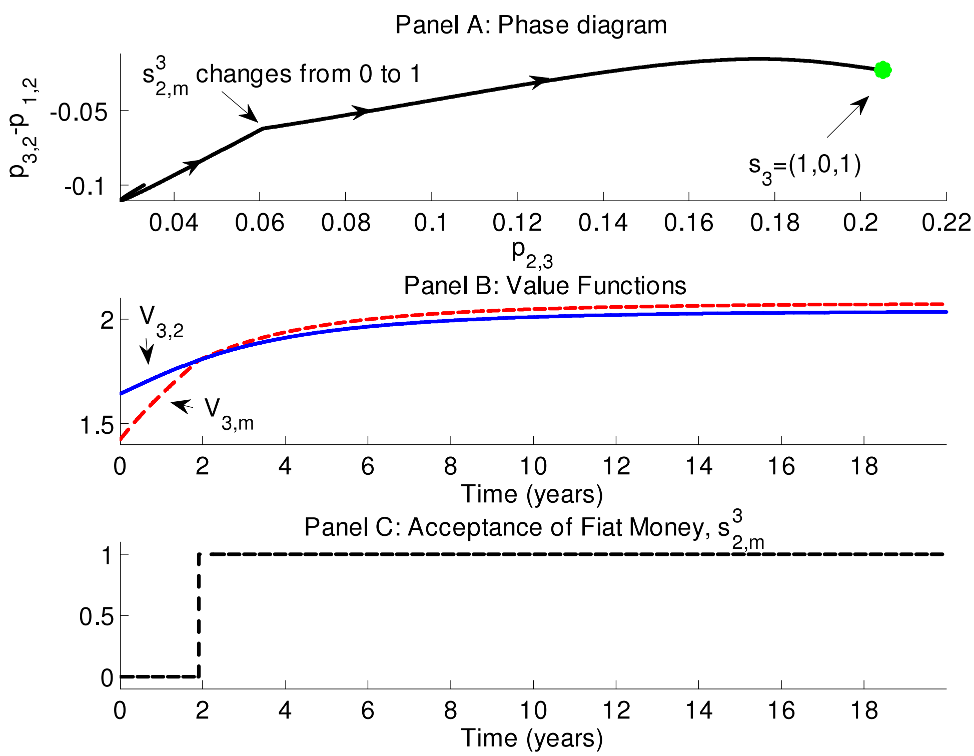

Fig. 7. Acceptance of commodity and Fiat Money, Model B. Note: The values of the parameters are: $\delta_m = 0.1$, $M = 0.3$, and $\theta_i = \frac{1}{3}$. See (Table 1), Model B, for remaining parameter values. The initial condition is $\mathbf{p}(0) = (\frac{1}{2}\theta_1, \frac{1}{10}\theta_2, \theta_3, M, 0)$.

$$\begin{aligned}\frac{d}{d\tau}\pi^i_{i+2,j}(\tau,t) =&\alpha\left\{\sum_{i'}\sum_k \pi^i_{k,j}p_{i',i+2}\sigma^i_{k,i+2}s^{i'}_{i+2,k}\right.\\ &\left.-\sum_{i'}\sum_k \pi^i_{i+2,j}p_{i',k}\sigma^i_{i+2,k}s^{i'}_{k,i+2}\right\} - \delta_g\pi^i_{i+2,j}\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{d}{d\tau}\pi^i_{m,j}(\tau,t) =&\alpha\left\{\sum_{i'}\sum_k \pi^i_{k,j}p_{i',m}\sigma^i_{k,m}s^{i'}_{j,m}\right.\\ &\left.-\sum_{i'}\sum_k \pi^i_{m,j}p_{i',k}\sigma^i_{m,k}s^{i'}_{k,m}\right\}\\ &-\delta_m\pi^i_{m,j} + \delta_g(\pi^i_{i+1,j} + \pi^i_{i+2,j})\end{aligned} \tag{24}$$

with initial condition $\pi^i_{k,j}(t,t) = 1$ if $k = j$, and 0 otherwise. For each $i = 1, 2, 3$, we consider the 3×3 matrix $\mathcal{A}^i = \{\mathcal{A}^i_{j,k}\}_{j,k=i+1,i+2,m}$ defined as:

$$\begin{aligned}
\mathcal{A}^i_{i+1,i+1} &= -\alpha\sum_{i'}\sum_{k\neq i+1} p_{i',k}\sigma^i_{i+1,k}s^{i'}_{k,i+1} - \delta_g\\
\mathcal{A}^i_{i+1,i+2} &= \alpha\sum_{i'} p_{i',i+2}\sigma^i_{i+1,i+2}s^{i'}_{i+2,i+1}\\
\mathcal{A}^i_{i+1,m} &= \alpha\sum_{i'} p_{i',m}\sigma^i_{i+1,m}s^{i'}_{m,i+1} + \delta_g\\
\\
\mathcal{A}^i_{i+2,i+1} &= \alpha\sum_{i'} p_{i',i+1}\sigma^i_{i+2,i+1}s^{i'}_{i+1,i+2} + \alpha\sum_{i'} p_{i',i}s^{i'}_{i,i+2}\\
\mathcal{A}^i_{i+2,i+2} &= -\alpha\sum_{i'}\sum_{k\neq i+2} p_{i',k}\sigma^i_{i+2,k}s^{i'}_{k,i+2} - \alpha\sum_{i'} p_{i',i}s^{i'}_{i,i+2} - \delta_g\\
\mathcal{A}^i_{i+2,m} &= \alpha\sum_{i'} p_{i',m}\sigma^i_{i+2,m}s^{i'}_{m,i+2} + \delta_g\\
\\
\mathcal{A}^i_{m,i+1} &= \alpha\sum_{i'} p_{i',i+1}\sigma^i_{m,i+1}s^{i'}_{i+1,m} + \alpha\sum_{i'} p_{i',i}s^{i'}_{i,m} + \delta_m\\
\mathcal{A}^i_{m,i+2} &= \alpha\sum_{i'} p_{i',i+2}\sigma^i_{m,i+2}s^{i'}_{i+2,m}\\
\mathcal{A}^i_{m,m} &= -\alpha\sum_{i'}\sum_{k\neq m} p_{i',k}\sigma^i_{m,k}s^{i'}_{k,m} - \alpha\sum_{i'} p_{i',i}s^{i'}_{i,m} - \delta_m\,.
\end{aligned} \tag{25}$$

The expressions in (22)–(24) then simplify to

$$\frac{d}{d\tau}\pi^i_{k,j}(\tau,t) = \sum_l \mathcal{A}^i_{l,k}(\tau)\pi^i_{l,j}(\tau,t).$$

Observe that the matrix $\mathcal{A}^i$ satisfies $\mathcal{A}^i_{j,k} \geq 0$ for $j \neq k$ and

$$\sum_l \mathcal{A}^i_{j,l} = 0\,, \tag{26}$$

for every j. These properties are used in proving Proposition 2 and Lemma 3 (see below).

Calling $\Pi^i(\tau,t)$ the 3×3 matrix with entries $(\Pi^i)_{k,j}(\tau,t) = \pi^i_{k,j}(\tau,t)$, the above equation can be written as

$$\frac{d}{d\tau}\Pi^i(\tau,t) = \mathcal{A}^i(\tau)^T\Pi^i(\tau,t) \tag{27}$$

where $\mathcal{A}^i(t)^T$ is the transpose of $\mathcal{A}^i(t)$. What is needed to compute the evolution of $V_{i,j}(t)$, however, is the derivative of $\Pi^i(\tau,t)$ with respect to t rather than with respect to τ. Note, however, that from (27) it follows that

$$\Pi^i(\tau,t-dt) = \Pi^i(\tau,t)\Pi^i(t,t-dt) = \Pi^i(\tau,t)\left(1 + dt\mathcal{A}^i(t)^T\right).$$

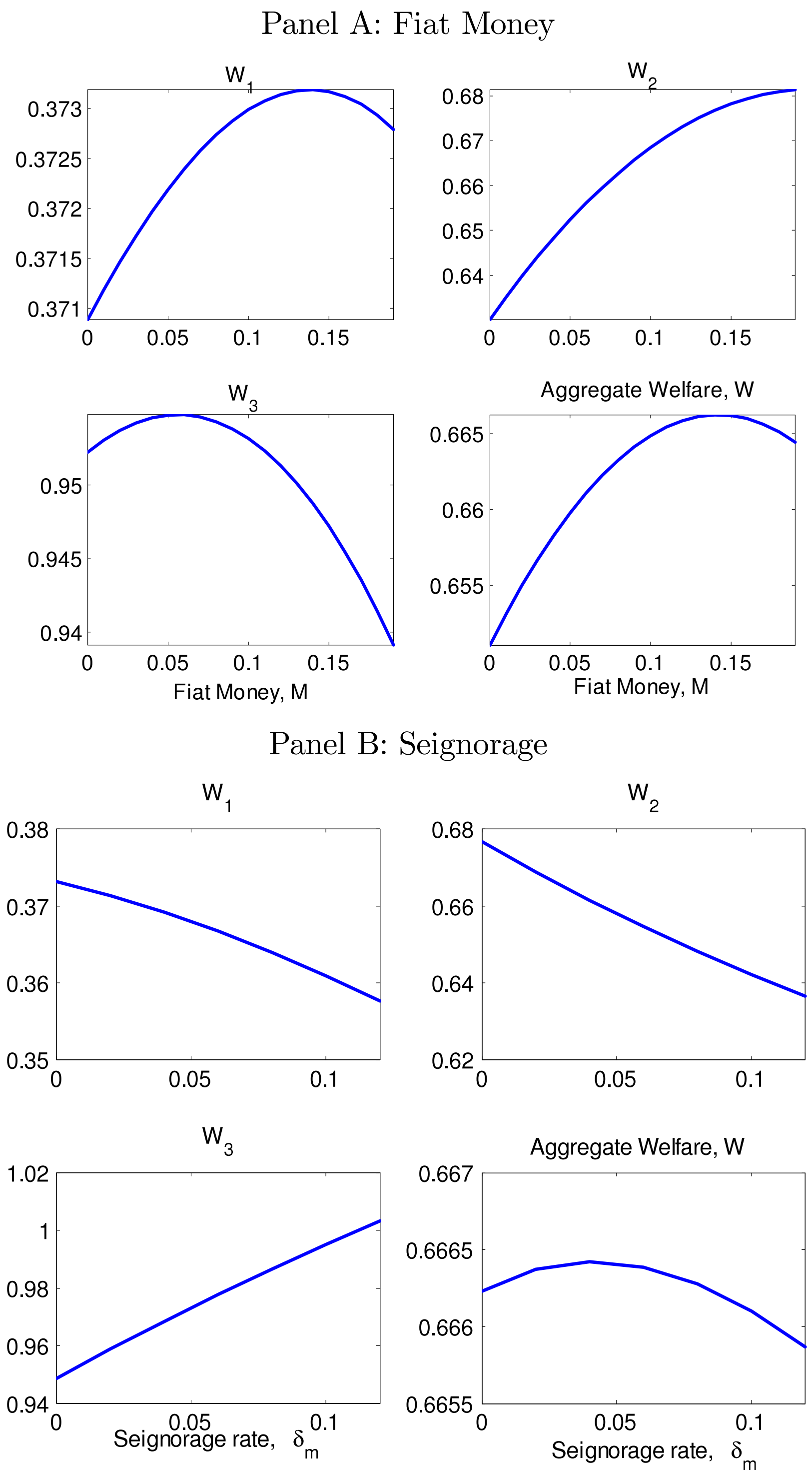

Fig. 8. Welfare, Fiat Money, and seignorage. Note. The values of δ_m and M that maximize the society's welfare are 0.04 and 0.14, respectively. The population is equally split between the three types ($\theta_i = \frac{1}{3}$). For remaining parameters see (Table 1), Model A.

Consequently,

$$\frac{d}{dt}\Pi^i(\tau,t) = -\Pi^i(\tau,t)\mathcal{A}^i(t)^T,$$

that in extended form becomes

$$\frac{d}{dt}\pi^i_{k,j}(\tau,t) = -\sum_l \mathcal{A}^i_{j,l}(t)\pi^i_{k,l}(\tau,t). \tag{28}$$

By inserting (28) in (20) we obtain

$$\dot{\mathbf{V}}_i = \rho\mathbf{V}_i - \mathcal{A}^i(t)\mathbf{V}_i - \mathbf{v}^i(t) \tag{29}$$

where $\mathbf{v}^i(t) = (v_{i,i+1}(\mathbf{p}(t)), v_{i,i+2}(\mathbf{p}(t)), v_{i,m}(\mathbf{p}(t)))$. Using the definition of the matrix $\mathcal{A}^i$ in (25) and of $v_{i,j}$ in (21) we get, for $j = i+1$

or $i+2$,

$$\begin{aligned}\dot{V}_{i,j} =&\rho V_{i,j} - \alpha \left\{ \sum_{i'} \sum_{k\neq i} p_{i',k}\sigma^i_{j,k} s^{i'}_{k,j} V_{i,k} + \sum_{i'} p_{i',i} s^{i'}_{i,j} V_{i,i+1} \right.\\ &\left. - \sum_{i',k} p_{i',k}\sigma^i_{j,k} s^{i'}_{k,j} V_{i,j} \right\}\\ &-\delta_g(V_{i,m}-V_{i,j}) - \sum_{i'} p_{i',i} s^{i'}_{j,i} u_i + c_j.\end{aligned}$$

Rearranging terms we get:

$$\begin{aligned}\dot{V}_{i,j} =&(\rho+\delta_g+\alpha)V_{i,j} - \alpha \left\{ \sum_{i'} \sum_{k\neq i} p_{i',k}\sigma^i_{j,k} s^{i'}_{k,j} V_{i,k} \right.\\ &+ \sum_{i'} p_{i',i} s^{i'}_{i,j}(V_{i,i+1}+u_i)\\ &\left. + \sum_{i',k} p_{i',k}(1-\sigma^i_{j,k} s^{i'}_{k,j}) V_{i,j} \right\} - \delta_g V_{i,m} + c_j.\end{aligned}$$

This is the expression in (7) when $j = i+1$ or $j+2$. Similarly, when $j = m$

$$\begin{aligned}\dot{V}_{i,m} =&\rho V_{i,j} - \alpha \left\{ \sum_{i'} \sum_{k\neq i} p_{i',k}\sigma^i_{m,k} s^{i'}_{k,m} V_{i,k} + \sum_{i'} p_{i',i} s^{i'}_{i,m} V_{i,i+1} \right.\\ &\left. + \sum_{i',k} \sigma^i_{m,k} s^{i'}_{k,m} V_{i,j} \right\}\\ &-\delta_m(V_{i,i+1}-V_{i,m}) - \sum_{i'} p_{i',i} s^{i'}_{i,m} u_i - \delta_m D_i\\ =&(\rho+\delta_g+\alpha)V_{i,m} - \alpha \left\{ \sum_{i'} \sum_{k\neq i} p_{i',k}\sigma^i_{m,k} s^{i'}_{k,m} V_{i,k} \right.\\ &+ \sum_{i'} p_{i',i} s^{i'}_{i,m}(V_{i,i+1}+u_i)\\ &\left. + \sum_{i',k}(1-\sigma^i_{m,k} s^{i'}_{k,m}) V_{i,j} \right\} + \delta_m(V_{i,i+1}-D_i).\end{aligned}$$

This is expression in (7) for $j = m$. ■

A.1. On the solution of value functions in (7)–(9).

For the proof of the remaining propositions we need a more explicit representation of $\mathbf{V}_i(t)$. Thus we will obtain an integral formula for the solution of (7)–(9) with the initial condition $\mathbf{W}$ at time T, that is, with the condition $\mathbf{V}_i(T) = \mathbf{W}$. To this end we start with the equivalent formulation contained in (29). This is a system of non-autonomous (since $\mathcal{A}^i(t)$ depends on t) and non-homogeneous (due to the $\mathbf{v}^i(t)$ term) ordinary linear differential equations.

To solve this system we first look for the fundamental solution of the associated homogeneous system, that is we consider the 3×3 matrix $\Phi^i(t,T)$ that solves the initial value problem

$$\begin{cases} \dfrac{d}{dt}\Phi^i(t,T) = -\mathcal{A}^i(t)\Phi(t,T) \\ \Phi^i(T,T) = \mathrm{Id}\ . \end{cases} \tag{30}$$

where $\mathcal{A}^i(t)$ is the matrix defined in (25) and Id is the 3×3 identity matrix. Further properties of $\Phi^i(t,T)$ will be studied in Lemma 2.

Writing $\mathbf{V}_i(t) = \Phi^i(t,T)\mathbf{Z}(t)$ we get the equation

$$\dot{\mathbf{Z}}(t) = \rho\mathbf{Z}(t) - \Phi^i(t,T)^{-1}\mathbf{v}^i(t)$$

that can be easily solved by

$$\mathbf{Z}(t) = e^{\rho(t-T)}\mathbf{Z}(T) - \int_T^t e^{\rho(t-\tau)}\Phi^i(\tau,T)^{-1}\mathbf{v}^i(\tau)d\tau.$$

Using that $\Phi^i(t,T)\Phi^i(\tau,T)^{-1} = \Phi^i(t,\tau)$ and observing that $\mathbf{Z}(T) = \mathbf{V}_i(T) = \mathbf{W}$ we finally get

$$\mathbf{V}_i(t) = e^{\rho(t-T)}\Phi^i(t,T)\mathbf{W} - \int_T^t e^{\rho(t-\tau)}\Phi^i(t,\tau)\mathbf{v}^i(\tau)d\tau. \tag{31}$$

Lemma 2. *Let $\Phi^i(t,T)$ be defined by (30). Then for every $t < T$ we have*

$$\Phi^i_{j,k}(t,T) \geq 0 \qquad \Phi^i_{j,j}(t,T) > 0 \tag{32}$$

and

$$\|\Phi^i(t,T)\|_\infty \leq 1 \qquad \forall t \leq T \tag{33}$$

where $\|\mathbf{U}_i\|_\infty = \sup_j U_{i,j}$.

Proof. Since $\mathcal{A}^i(t)$ is continuous in t, we can use Euler approximations for the solution of (30). This gives

$$\Phi(t,T) = \lim_{N\to\infty} \prod_{k=1}^N (\mathrm{Id} - \delta t\mathcal{A}(t_k))\,\Phi(T,T)$$

where

$$\delta t = \frac{t-T}{N} \qquad t_k = T + k\delta t.$$

Observe that $\delta t < 0$ since $t < T$. Let $\mathcal{C}(t_k) = \mathrm{Id} - \delta t\mathcal{A}(t_k)$. For δt small enough, $\mathcal{C}_{l,j}(t_k) \geq 0$, $\mathcal{C}_{l,l}(t_k) > 0$ for every l, j and

$$\sum_j \mathcal{C}_{l,j}(t_k) = 1 \qquad \forall l.$$

This implies (32). Moreover we have

$$\|\mathcal{C}(t_k)\mathbf{W}\|_\infty = \sup_l \left|\sum_j \mathcal{C}_{l,j}W_j\right| \leq \sup_l \sup_j |W_j| \sum_j |\mathcal{C}_{l,j}| = \|\mathbf{W}\|_\infty\ .$$

This gives

$$\|\Phi(t,T)\|_\infty \leq \lim_{N\to\infty} \prod_{k=N}^1 \|\mathcal{C}(t_k)\|_\infty \leq 1. \quad ■$$

Proof of Proposition 2. Consider a variation of $\sigma^i_{i+1,i+2}(t)$ of the form

$$\tilde{\sigma}^i_{i+1,i+2}(t) = \sigma^i_{i+1,i+2}(t) + \epsilon\eta(t).$$

where $\eta(t) = 0$ for $t > T$ and ϵ is a parameter. Differentiating (29) with respect to ϵ delivers

$$\partial_\epsilon\dot{\mathbf{V}}^i(t) = \rho\partial_\delta\mathbf{V}_i - \mathcal{A}^i(t)\partial_\epsilon\mathbf{V}_i - \partial_\epsilon\mathcal{A}^i(t)\mathbf{V}_i.$$

This system of equations is of the same form as (29) with $\partial_\epsilon\mathbf{V}$ as the unknown and $\partial_\epsilon\mathcal{A}^i(t)\mathbf{V}_i$ as the non homogeneous term. Following the same procedure that led to (31) we obtain

$$\partial_\epsilon\mathbf{V}_i(t) = \int_t^T e^{-\rho(\tau-t)}\Phi(t,\tau)\left(\partial_\epsilon\mathcal{A}^i(\tau)\right)\mathbf{V}_i(\tau)d\tau.$$

where we used that $\partial_\epsilon\mathbf{V}_i(t) = 0$ if $t \geq T$. From (25) it follows $\partial_\epsilon\mathcal{A}^i_{i,k}(\tau) \neq 0$ only if $j = i+1$ and $k = i+1$ or $i+2$. Moreover

$$\partial_\epsilon\mathcal{A}^i_{i,i+1}(\tau) = -\partial_\epsilon\mathcal{A}^i_{i,i+2}(\tau) = \alpha\eta(\tau)\sum_{i'} p_{i',i+2}(\tau)s^{i'}_{i+2,i+1}(\tau)$$

so that

$$\partial_\epsilon \mathcal{A}^i(\tau)\mathbf{V}_i(\tau)\big|_{\epsilon=0} = -\alpha\eta(\tau)\Delta^i_{i+1,i+2}(\tau) \times \begin{pmatrix} \sum_{i'} p_{i',i+2}(\tau)s^{i'}_{i+2,i+1}(\tau) \\ 0 \\ 0 \end{pmatrix}.$$

Therefore,

$$\partial_\epsilon \mathbf{V}_i(t)\big|_{\epsilon=0} = \int_t^\infty \eta(\tau)\Delta^i_{i+1,i+2}(\tau)\mathbf{U}(t,\tau)d\tau,$$

where

$$\mathbf{U}(t,\tau) = e^{-\rho(\tau-t)}\Phi(t,\tau)\begin{pmatrix} \sum_{i'} p_{i',i+2}(\tau)s^{i'}_{i+2,i+1}(\tau) \\ 0 \\ 0 \end{pmatrix}.$$

From (32) and the fact that $p_{i+1,i+2}(\tau) > 0$ and $s^{i+1}_{i+2,i+1}(\tau) = 1$, it follows that $\mathbf{U}(t,\tau)_{i+1} > 0$ and that $\mathbf{U}(t,\tau)_j \geq 0$ for $j = i+2, m$.

Clearly, if $\Delta^i_{i+1,i+2}(\tau) \neq 0$, the contribution of $\eta(\tau)$ to the variation $\partial_\delta \mathbf{V}_i(t) > 0$ is different from zero and there is no critical value for $\sigma^1_{i+1,i+2}(\tau) \in (0,1)$. We can then conclude $\mathbf{V}_i(t)$ reaches a maximum at a boundary, i.e. $\sigma^1_{i+1,i+2}(\tau) \in \{0,1\}$, and that

$$\sigma^i_{i+1,i+2}(\tau) = \begin{cases} 1 & \Delta^i_{i+1,i+2}(\tau) < 0 \\ 0 & \Delta^i_{i+1,i+2}(\tau) > 0\,. \end{cases} \tag{34}$$

Finally, as already observed in footnote 8 after Proposition 2, since $\Delta^i_{i+1,i+2}(t) = 0$ for a finite set of switching times, the value of $\sigma^i_{i+1,i+2}(t)$ on such a set does not affect $\mathbf{V}_i(t)$. Similar observations hold for $\sigma^i_{j,k}$ with $(j,k) \neq (i+1, i+2)$. This concludes the proof of (11). ∎

Proof of Proposition 4. Since $\mathbf{p}^*$ is a stable fixed point, for ϵ sufficiently small, we can find C_ϵ such that

$$\|\mathbf{p}(0) - \mathbf{p}^*\| \leq C_\epsilon \quad \text{implies} \quad \|\mathbf{p}(t) - \mathbf{p}^*\| \leq \epsilon \qquad \forall t > 0\,.$$

Since $\mathbf{p}^*$ is a steady state, $\mathbf{V}^*_i$ must be a fixed point of (29), that is

$$\rho\mathbf{V}^*_i - \mathcal{A}^{i,*}\mathbf{V}^*_i - \mathbf{v}^{i,*} = 0 \tag{35}$$

where $\mathcal{A}^{i,*}$ and $\mathbf{v}^*$ are defined in (21) and (25) with $\mathbf{p} = \mathbf{p}^*$. Calling $\delta\mathbf{V}_i(t) = \mathbf{V}_i(t) - \mathbf{V}^*$, by subtracting (35) from (29) we get

$$\delta\dot{\mathbf{V}}_i = \rho\delta\mathbf{V}_i - \mathcal{A}^i(t)\delta\mathbf{V}_i - \mathbf{w}(t)$$

where

$$\mathbf{w}(t) = \left(\mathcal{A}^i(t) - \mathcal{A}^{i,*}\right)\mathbf{V}^*_i + (\mathbf{v}^i(t) - \mathbf{v}^{i,*}).$$

This is a system of linear differential equations of the form of (29). Thus we can write its solution as

$$\delta\mathbf{V}_i(t) = \mathbf{V}_i(t) - \mathbf{V}^*_i = \int_\infty^t e^{\rho(t-s)}\Phi(t,s)\mathbf{w}(s)ds. \tag{36}$$

Since $\mathcal{A}^i(t)$ and $\mathbf{v}^i(t)$ are smooth functions of $\mathbf{p}(t)$, we can find a constant K such that $\|\mathbf{w}(t)\| \leq K\epsilon$ when $\|\mathbf{p}(t) - \mathbf{p}^*\| \leq \epsilon$. From (33) and (36) we get that $\|\mathbf{V}_i(t) - \mathbf{V}^*_i\|_\infty \leq K\epsilon/\rho$ uniformly in t. Since $(\mathbf{p}^*, \mathbf{s}^*)$ forms a Nash steady state, $\mathbf{s}^*$ and $\mathbf{V}^*$ must satisfy (11). It follows that $\mathbf{V}_i(t)$ and $\mathbf{s}^*$ still satisfy (11) if ϵ is small enough. Thus, if $\|\mathbf{p}(0) - \mathbf{p}^*\|$ is small enough, $\mathbf{s}^*$ is the best response to itself for all $t > 0$. It follows that $(\mathbf{p}(t), \mathbf{s}^*)$ is a Nash equilibrium. Observe that while the proof relies on a simple continuity statement for the value functions $\mathbf{V}_i$, the result is significant, for it clears up an important potential hurdle for the analysis of the dynamical system. Renero (1998) showed that in this model with no fiat money, one can find a mixed strategy steady state Nash equilibrium $(\mathbf{p}^*, \mathbf{s}^*)$ asymptotically stable for (1)–(3) and that nevertheless there is no mixed strategy Nash equilibrium that starts near $\mathbf{p}^*$ and converges to $\mathbf{p}^*$. ∎

Proof of Proposition 5. To prove the proposition, it is useful to state the stability properties of (7). Let $\mathbf{V}_i(t, T, \mathbf{W})$ be the solution of (7) obtained by setting $\mathbf{V}_i(T, T, \mathbf{W}) = \mathbf{W}$.

Lemma 3. *Given a set of strategies $\mathbf{s}(t)$ and $\boldsymbol{\sigma}(t)$, and a pattern $\mathbf{p}(t)$, for any $\mathbf{W}_1$ and $\mathbf{W}_2$:*

$$\|\mathbf{V}_i(t,T,\mathbf{W}_1) - \mathbf{V}_i(t,T,\mathbf{W}_2)\|_\infty \leq e^{-\rho(T-t)}\|\mathbf{W}_1 - \mathbf{W}_2\|_\infty \tag{37}$$

where $t \leq T$. Thus the value function at time t of agent a_i can be computed as

$$\mathbf{V}_i(t) = \lim_{T\to\infty} \mathbf{V}_i(t,T,\mathbf{W}) \tag{38}$$

where the limit does not depend on $\mathbf{W}$ and it is reached exponentially.

Proof of Lemma 3. From (31) we have

$$\mathbf{V}_i(t,T,\mathbf{W}) = e^{\rho(t-T)}\Phi(t,T)\mathbf{W} - \int_T^t e^{\rho(t-\tau)}\Phi(t,\tau)\mathbf{v}(\tau)d\tau, \tag{39}$$

so that

$$\mathbf{V}_i(t,T,\mathbf{W}_1) - \mathbf{V}_i(t,T,\mathbf{W}_2) = e^{\rho(t-T)}\Phi(t,T)(\mathbf{W}_1 - \mathbf{W}_2)\,.$$

This implies

$$\|\mathbf{V}_i(t,T,\mathbf{W}_1) - \mathbf{V}_i(t,T,\mathbf{W}_2)\|_\infty = e^{\rho(t-T)}\|\Phi(t,T)\|_\infty\|\mathbf{W}_1 - \mathbf{W}_2\|_\infty$$

The thesis follows from Lemma 2. ∎

Returning to Proposition 5, observe that $\mathbf{V}_i(t) = \mathbf{V}_i(t,T,\mathbf{V}_i(T))$. From Lemma 3 we get

$$\|\mathbf{V}_i(t) - \mathbf{V}_i(t,T,\mathbf{V}^*_i)\|_\infty \leq e^{-\rho(T-t)}\|\mathbf{V}_i(T) - \mathbf{V}^*_i\|_\infty.$$

Finally, to go from the infinity distance $\|\mathbf{V}_i(t) - \mathbf{V}_i(t,T,\mathbf{V}^*_i)\|_\infty$ to euclidean distance $\|\mathbf{V}_i(t) - \mathbf{V}_i(t,T,\mathbf{V}^*_i)\|$ we observe that for every $\mathbf{W}$ we have $\|\mathbf{W}\|_\infty \leq \|\mathbf{W}\| \leq \sqrt{3}\|\mathbf{W}\|_\infty$. The expression in Proposition 5 can thus be obtained as

$$\begin{aligned} \|\mathbf{V}_i(t) - \mathbf{V}_i(t,T,\mathbf{V}^*_i)\| &\leq \sqrt{3}\|\mathbf{V}_i(t) - \mathbf{V}_i(t,T,\mathbf{V}^*_i)\|_\infty \\ &\leq \sqrt{3}e^{-\rho(T-t)}\|\mathbf{V}_i(T) - \mathbf{V}^*_i\|_\infty \\ &\leq \sqrt{3}e^{-\rho(T-t)}\|\mathbf{V}_i(T) - \mathbf{V}^*_i\|\,. \end{aligned}$$ ∎

Appendix B. Nash equilibria and the stable manifold theorem

The iteration procedure to find Nash equilibria is similar to that used by Perron to prove the stable manifold theorem (for an illustration see, among others, Robinson (1995)). Here, we discuss similarities and differences. Consider the system of differential equations

$$\begin{cases} \dot{x} = -\lambda x + f^-(x,y) \\ \dot{y} = \mu y + f^+(x,y) \end{cases} \tag{40}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\lambda, \mu > 0$ and

$$\lim_{|x|+|y|\to 0} \frac{|f^-(x,y)| + |f^+(x,y)|}{|x| + |y|} = 0.$$

The system (40) is the sum of linear ($-\lambda x$ and μy) and non-linear terms (f^+ and f^-); its fixed point is $(x,y) = (0,0)$.

The stable manifold theorem states that for every $x_0 \in \mathbb{R}^n$, with $|x_0|$ sufficiently small, there is unique y_0 such that the solution $(x(t), y(t))$ of (40) starting at (x_0, y_0) satisfies

$$\lim_{t\to\infty}(x(t), y(t)) = (0,0). \tag{41}$$

Moreover, it says that the point y_0 is given by a smooth function of x_0, that is $y_0 = W^{-}(x_0)$. The graph of W^{-}, that is, the set $(x_0, W^{-}(x_0))$, is called the local stable manifold of $(0, 0)$. Perron's proof is based on the following representation of the solution of (40):

$$x(t) = e^{-\lambda t}x_0 + \int_0^t e^{-\lambda(t-\tau)}f^{-}(x(\tau), y(\tau))d\tau \tag{42}$$

$$y(t) = \int_{\infty}^t e^{\mu(t-\tau)}f^{+}(x(\tau), y(\tau))d\tau. \tag{43}$$

The proof starts from a guess $x^0(t) = x_0e^{-\lambda t}$ and $y^0(t) = 0$ for all $t > 0$. It then computes a new approximation for the evolution of the stable variable $x(t)$, $x^1(t)$, through (42). Inserting x^1 and y^0 into (43) yields an approximation for the unstable variable $y^1(t)$. Note that while in (42) time runs forward (τ goes from 0 to t), in (43) it runs backward (in τ goes from ∞ to t). Therefore both integrations are stable. Iterating these two steps yields a sequence $(x^n(t), y^n(t))$ that approximates the solution (42)–(43). Because the exponential factors in the integrals of (42) and (43) have negative exponents, if x_0 is sufficiently small, the map from (x^n, y^n) to (x^{n+1}, y^{n+1}) is a contraction. Finally, the Banach fixed-point theorem guarantees that the sequence $(x^n(t), y^n(t))$ converges uniformly to a solution $(x(t), y(t))$ of (40) with $x(0) = x_0$ and satisfying (41).

There are similarities between Perron's approach in proving the manifold theorem and the construction of Nash equilibria discussed in Section 4. First, the Nash steady state $(\mathbf{p}^*, \mathbf{s}^*)$ corresponds to the fixed point $(0, 0)$ of (40). Second, the $\mathbf{p}$ in (1)–(3) is comparable to the x in (40). Third, in (39) the discount rate ρ plays the same role of the unstable exponent μ in (43). Fourth, the value function $\mathbf{V}^i$ in (7) is somewhat comparable to y in (40). But there is also an important difference: The value function $\mathbf{V}^i$ affects the evolution of $\mathbf{p}$ through the intermediation of the strategies $\mathbf{s}$. Furthermore, in contrast to (40), the procedure we presented in Section 4 does not split the evolution of $(\mathbf{p}(t), \mathbf{s}(t))$ near $(\mathbf{p}^*, \mathbf{s}^*)$ between a linear and a nonlinear part. Therefore, it is better suited to follow the dynamics away from the steady state.

As noted in Section 4, it is important to recognize that the evolution of the distribution of inventories, $\mathbf{p}(t)$, and that of the value functions $\mathbf{V}(t)$, can be studied jointly. Starting from a $\mathbf{p}(T)$ close to the steady state $\mathbf{p}^*$, with a $\mathbf{V}_i(T) = \mathbf{V}_i^*$ and $\boldsymbol{\sigma}_i(T) = \boldsymbol{\sigma}_i^*$, one may integrate (7) and (1)–(3) backward in time $- t$ goes from T to 0. To find the approximate solution of the Nash equilibrium it would suffice to alter $\boldsymbol{\sigma}_i(t)$ along the integration process so as to be consistent with the value functions $\mathbf{V}_i(t)$, i.e., to satisfy (11). While this procedure, sometime called *backward propagation*, usually works well for low-dimensional systems (see, for instance, Brunner and Strulik (2002)), it presents limitations for the type of research question that we address. Our objective is to obtain a pattern $(\mathbf{s}(t), \mathbf{p}(t))$ that goes through any initial conditions, that is, through any arbitrary points in the space of the distribution of inventories, $\mathbf{p}(0)$. As the dimension of the manifold expands, guiding the system toward a particular point on the state space by integrating (7) and (1)–(3) backward in time becomes challenging because some regions of the manifold may be hard to reach. Conversely, the method proposed here offers total control over the initial condition, an indispensable feature for running macroeconomic experiments.

Our algorithm presents some similarities with the *policy iteration* algorithm. A policy iteration algorithm would start with a guess of $\mathbf{s}(\mathbf{p})$ given for every initial distribution of stocks $\mathbf{p}$. Next it would compute the best response $\sigma(\mathbf{p})$, again for every distribution of stocks $\mathbf{p}$, as described in Section 4.[14] By iterating the procedure, it would yield a Markov Perfect Nash Equilibrium, if convergence is attained (see, for example, Maskin and Tirole (2001)). Observe that this algorithm would be numerically prohibitive in our situation. The main difference with the algorithm described in Section 4 is that we study σ as a function of time t instead of stock $\mathbf{p}$. In this respect our algorithm can be seen as an efficient hybrid of the *backward propagation* and the *policy iteration* algorithms.

[14] For a system with a finite number of states, the best response is the solution of a linear system of equations. The situation is quite different in our case.

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