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Quantum Phase Transition in an Interacting Fermionic Chain

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Abstract. We rigorously analyze the quantum phase transition between a metallic and an insulating phase in (non-solvable) interacting spin chains or one-dimensional fermionic systems. In particular, we prove the persistence of Luttinger liquid behavior in the presence of an interaction even arbitrarily close to the critical point, where the Fermi velocity vanishes and the two Fermi points coalesce. The analysis is based on two different multiscale analysis; the analysis of the first regime provides gain factors which compensate the small divisors due to the vanishing Fermi velocity.

1. Introduction

1.1. Spin or Fermionic Chains

Recently a great deal of attention has been focused on the quantum phase transition between a metallic and an insulating phase in (non-solvable) interacting spin chains or one-dimensional fermionic systems. Besides its intrinsic interest, such problem has a paradigmatic character, see e.g., [1,2]. Interacting fermionic systems are often investigated using bosonization [3], but such method cannot be used in this case; it requires linear dispersion relation, while in our case close to the critical point the dispersion relation becomes quadratic. Interacting fermionic systems with non-linear dispersion relation have been studied using convergent expansions, based on rigorous Renormalization Group methods. However, the estimate for the radius of convergence of the expansions involved vanishes at the critical point, so that they provide no information close to the quantum phase transition. This paper contains the first rigorous study of the behavior close to the metal insulator transition, using an expansion convergent uniformly in a region of parameters including the critical point.

We will focus for definiteness on the model whose Hamiltonian is given by

$$H = -\sum_x \frac{1}{2}\left[S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2\right] - \lambda \sum_{x,y} v(x-y)S_x^3 S_y^3 - \bar{h}\sum_x S_x^3 + U_L \quad (1)$$

where $(S_x^1, S_x^2, S_x^3) = \frac{1}{2}\left(\sigma_x^1, \sigma_x^2, \sigma_x^3\right)$, for $x = 1, 2, \ldots, L$, σ_x^i, $i = 1, 2, 3$ are Pauli matrices, $\bar{h}$ is the magnetic field and $v(x)$ is a short range even potential, that is

$$v(-x) = v(x), \qquad |v(x)| \le e^{-\kappa|x|}.$$

Finally U_L is an operator depending only on S_1^i and S_L^i to be used later to fix the boundary conditions. When $v(x-y) = \delta_{x,y+1}$, this model is known as XXZ Heisenberg spin chain. Setting $\mathbf{x} = (x_0, x)$, we define $S_{\mathbf{x}}^i = e^{Hx_0}S_x^i e^{-Hx_0}$. Moreover, given an observable O, we define

$$\langle O \rangle_{\beta,L} = \frac{\mathrm{Tr}\, e^{-\beta H} O}{\mathrm{Tr}\, e^{-\beta H}} \quad \text{and} \quad \langle O \rangle = \lim_{\beta\to\infty}\lim_{L\to\infty} \langle O \rangle_{\beta,L}. \quad (2)$$

It is well known that spin chains can be rewritten in terms of fermionic operators $a_x^\pm$, with $\{a_x^+, a_y^-\} = \delta_{x,y}$, $\{a_x^+, a_y^+\} = \{a_x^-, a_y^-\} = 0$, by the *Jordan–Wigner transformation*:

$$\sigma_x^- = e^{-i\pi \sum_{y=1}^{x-1} a_y^+ a_y^-} a_x^-, \quad \sigma_x^+ = a_x^+ e^{i\pi \sum_{y=1}^{x-1} a_y^+ a_y^-}, \quad \sigma_x^3 = 2a_x^+ a_x^- - 1 \quad (3)$$

where $\sigma_x^\pm = (\sigma_x^1 \pm i\sigma_x^2)/2$. In terms of the fermionic operators the Hamiltonian becomes

$$H = -\sum_x \left[\frac{1}{2}\left(a_{x+1}^+ a_x^- + a_x^+ a_{x+1}^-\right) + ha_x^+ a_x^-\right] - \lambda\sum_{x,y} v(x-y)a_x^+ a_x^- a_y^+ a_y^- \quad (4)$$

where $h = \bar{h} - \lambda\hat{v}(0)$ and U_L can be chosen so to obtain periodic boundary conditions for the fermions, *i.e.*, $a_L^\pm = a_1^\pm$. Therefore, the spin chain (1) can be equivalently represented as a model for interacting spinless fermions in one dimension with *chemical potential* $\mu = -h$.

The 2-point Schwinger function is defined as

$$S_{L,\beta}(\mathbf{x} - \mathbf{y}) = \langle \mathbf{T}a_{\mathbf{x}}^- a_{\mathbf{y}}^+ \rangle_{L,\beta} \quad (5)$$

where $\mathbf{T}$ is the *time ordering* operator, that is $\mathbf{T}(a_{\mathbf{x}}^- a_{\mathbf{y}}^+) = a_{\mathbf{x}}^- a_{\mathbf{y}}^+$ if $x_0 > y_0$ and $\mathbf{T}(a_{\mathbf{x}}^- a_{\mathbf{y}}^+) = -a_{\mathbf{y}}^+ a_{\mathbf{x}}^-$ if $x_0 \le y_0$. We will mostly study the infinite volume zero temperature 2-points Schwinger function given by

$$\lim_{\beta\to\infty}\lim_{L\to\infty} S_{L,\beta}(\mathbf{x} - \mathbf{y}) = S(\mathbf{x} - \mathbf{y}).$$

1.2. Quantum Phase Transition in the Non-Interacting Case

The fermionic representation makes the analysis of the $\lambda = 0$ case (the so-called XX chain) quite immediate. We associate to the set of creation and annihilation operators the corresponding set of operators in momentum space writing

$$a_x^\pm = \frac{1}{L}\sum_{k\in\tilde{\mathscr{D}}} e^{\pm ikx}\hat{a}_k^\pm$$

where $\widetilde{\mathscr{D}} = \left\{ k \,|\, k = \frac{2\pi m}{L}, \ -\pi \le k < \pi \right\}$ and $\hat{a}_k^{\pm}$ are creation and annihilation operators verifying $\{\hat{a}_k^+, a_{k'}^+\} = \{\hat{a}_k^-, \hat{a}_{k'}^-\} = 0$, $\{\hat{a}_k^+, \hat{a}_k^-\} = L\delta_{k,k'}$ with δ the periodic Kronecker delta function. From these definitions we get

$$H_0 = \frac{1}{L} \sum_{k \in \widetilde{\mathscr{D}}} \varepsilon(k) \hat{a}_k^+ \hat{a}_k^- \qquad \varepsilon(k) = -\cos k - h \qquad (6)$$

where we have used the identity $\sum_{k \in \widetilde{\mathscr{D}}} \mathrm{e}^{ikx} = L\delta_{k,0}$. The *ground state* of (6) depends critically on h. Indeed, for $h < -1$ the ground state is the fermionic vacuum (empty band *insulating* state), for $h > 1$ it is the state with all fermionic levels occupied (filled band *insulating* state) while for $-1 < h < 1$ the ground state corresponds to the state in which all the fermionic levels with momenta $|k| \le p_F = \arccos(-h)$, are occupied (*metallic* state). p_F is called *Fermi momentum* and $\pm p_F$ are the *Fermi points* (the analogous of the Fermi surface in one dimension). In other words the values $h = \pm 1$ separate two different behaviors at zero temperature; one says that in correspondence of $h = \pm 1$ there is a *quantum phase transition* [1] between a metallic and an insulating phase.

The metallic or insulating phases are signaled by different properties of the 2-point Schwinger function, which is given by

$$S_{0,L,\beta}(\mathbf{x}) = \frac{1}{L} \sum_{k \in \widetilde{\mathscr{D}}} \mathrm{e}^{-ikx} \left\{ \frac{\mathrm{e}^{-x_0 \varepsilon(k)}}{1 + \mathrm{e}^{-\beta \varepsilon(k)}} \vartheta(x_0) - \frac{\mathrm{e}^{-(\beta + x_0)\varepsilon(k)}}{1 + \mathrm{e}^{-\beta \varepsilon(k)}} (1 - \vartheta(x_0)) \right\} \quad (7)$$

where $\vartheta(x_0) = 1$ if $x_0 > 0$ and $\vartheta(x_0) = 0$ otherwise. The Schwinger function (7) is defined for $-\beta \le x_0 \le \beta$ but it can be extended periodically over the whole real axis; such extension is smooth in x_0 for $x_0 \ne n\beta$, $n \in \mathbb{Z}$. It is easy to see that $S_{0,L,\beta}(n\beta^+, x) = S_{0,L,\beta}(n\beta^-, x)$ for $x \ne 0$ so that it is discontinuous only at $\mathbf{x} = (n\beta, 0)$. Since $S_{0,L,\beta}(\mathbf{x})$ is antiperiodic in x_0, it can be written in Fourier series except that at the discontinuity points; that is, for $\mathbf{x} \ne (n\beta, 0)$

$$S_{0,L,\beta}(\mathbf{x}) = \frac{1}{\beta L} \sum_{\mathbf{k} \in \mathscr{D}} \mathrm{e}^{-i\mathbf{k}\mathbf{x}} \hat{S}_{0,\beta,L}(\mathbf{k}) \qquad (8)$$

with $\mathbf{k} = (k_0, k)$, $\mathscr{D} = \left\{ \mathbf{k} \,|\, k = \frac{2\pi m}{L}, \ -\pi \le k < \pi, \ k_0 = \frac{2\pi}{\beta}\left(n + \frac{1}{2}\right) \right\}$ and

$$\hat{S}_{0,L,\beta}(\mathbf{k}) = \frac{1}{-ik_0 + \cos k + h}. \qquad (9)$$

In the metallic phase the Schwinger function $\hat{S}_0(\mathbf{k})$ is singular in correspondence of the Fermi points $(0, \pm p_F)$. For $|k|$ close to p_F we have $\hat{S}_0(\mathbf{k}) \sim \frac{1}{-ik_0 + v_F(|k| - p_F)}$. Notice that the 2-point Schwinger function is asymptotically identical, if the momenta are measured from the Fermi points, to the Schwinger function of massless Dirac fermions in $d = 1 + 1$ with Fermi velocity v_F. For values of h close to $h = -1$ (i.e., for small positive r if we set $h = -1 + r$) both the distance of the Fermi points and v_F are $O(\sqrt{r})$, that is the Fermi velocity vanishes with continuity and the two Fermi points coalesce. At criticality when $r = 0$ the 2-point function $\hat{S}_0(\mathbf{k})$ is singular only at $(0,0)$ and $\hat{S}_0(\mathbf{k}) \sim \frac{1}{-ik_0 + \frac{1}{2}k^2}$; the elementary excitations do not have a relativistic linear

dispersion relation, as in the metallic phase, but a parabolic one. Finally in the insulating phase for $r < 0$ the two point function has no singularities.

It is natural to ask what happens to the quantum phase transition in presence of the interaction.

1.3. Quantum Phase Transition in the Interacting Case

The Schwinger functions of the interacting model in the metallic phase have been constructed using Renormalization Group (RG) methods in [4–6,8,9]. Luttinger liquid behavior (in the sense of [7]) has been established, showing that the power law decay of correlations is modified by the interaction via the appearance of critical exponents, that depend in a non-trivial way on the interaction. It should be stressed that such analysis provides a full understanding inside the metallic phase, but gives no information on the phase transition; the reason is that the physical observables are expressed in terms of renormalized expansions which are convergent under the condition

$$|\lambda| \leq \varepsilon |v_F| \tag{10}$$

and small ε; therefore, the closer one is to the bottom (or the top) of the band, the smaller the interaction has to be chosen. This is not surprising, as such RG methods essentially show that the interacting fermionic chain is asymptotic to a system of interacting massless Dirac fermions in $d = 1 + 1$ dimensions with coupling $\frac{\lambda}{v_F}$. One may even suspect that an extremely weak interaction could produce some quantum instability close to the boundary of the metallic phase, where the parameters correspond to a strong coupling regime in the effective description.

This is, however, excluded by our results; we prove the persistence of the metallic phase, with Luttinger liquid behavior, in presence of interaction even arbitrarily close to the critical point, where the Fermi velocity vanishes. This result is achieved writing the correlations in terms of a renormalized expansion with a radius of convergence which is independent from the Fermi velocity. To obtain this result we needs to exploit the non-linear corrections to the dispersion relation due to the lattice. The proof is indeed based on two different multiscale analysis in two regions of the energy momentum space; in the smaller energy region the effective relativistic description is valid while in the larger energy region the quadratic corrections due to the lattice are dominating. The scaling dimensions in the two regimes are different; after the integration of the first regime one gets gain factors which compensate exactly the velocities at the denominator produced in the second regime, so that uniformity is achieved.

Our main results are summarized by the following theorem. We state it in terms of the Fourier transform of the 2-points Schwinger function defined by

$$\hat{S}_{L,\beta}(\mathbf{k}) = \int_0^\beta \mathrm{d}x_0 \sum_x \mathrm{e}^{i\mathbf{k}\mathbf{x}} S_{L,\beta}(\mathbf{x}) \tag{11}$$

for $\mathbf{k} \in \mathscr{D}$.

Theorem 1.1. *Given the Hamiltonian* (1) *with* $h = -1 + r$ *with* $|r| < 1$, *there exists* $\varepsilon > 0$ *and* $C > 0$ *(independent from* L, β, r) *such that, if* $|\lambda| < \varepsilon$ *then the Fourier transform of* $S_{L,\beta}(\mathbf{x})$ (5) *defined in* (11) *can be written in the following way.*

1. *For* $r > 0$ (*metallic phase*),

$$\hat{S}_{L,\beta}(\mathbf{k}) = \frac{\left[k_0^2 + \alpha(\lambda)^2(\cos k - 1 + \nu(\lambda))^2\right]^{\eta(\lambda)}}{-ik_0 + \alpha(\lambda)(\cos k - 1 + \nu(\lambda))}(1 + \lambda R_S(\lambda, \mathbf{k})) \qquad (12)$$

   where

$$\nu(\lambda) = r + \lambda r R_\nu(\lambda) \qquad \alpha(\lambda) = 1 + \lambda R_\alpha(\lambda)$$
$$\eta(\lambda) = b\lambda^2 r + \lambda^3 r^{\frac{3}{2}} R_\eta(\lambda) \qquad\qquad (13)$$

   with $b > 0$ *a constant and* $|R_i| \leq C$ *for* $i = S, \nu, \alpha$ *and* η.

2. *For* $r = 0$ (*critical point*)

$$\hat{S}_{L,\beta}(\mathbf{k}) = \frac{1 + \lambda R_S(\lambda, \mathbf{k})}{-ik_0 + \alpha(\lambda)(\cos(k) - 1)} \qquad (14)$$

   where $\alpha(\lambda) = 1 + \lambda R_\alpha(\lambda)$ *and* $|R_i| \leq C$ *for* $i = \alpha, S$.

3. *For* $r < 0$ (*insulating phase*)

$$|\hat{S}_{L,\beta}(\mathbf{k})| \leq \frac{C}{|r|}. \qquad (15)$$

Moreover, $\hat{S}(\mathbf{k}) = \lim_{\beta \to \infty} \lim_{L \to \infty} \hat{S}_{L,\beta}(\mathbf{k})$ *exists and is reached uniformly in* λ.

Clearly, by symmetry, similar results hold at the top of the band by setting $h = 1 - r$.

From the above result we see that in the metallic phase Luttinger liquid behavior is present; indeed the interaction changes the location of the Fermi points from $p_F = \pm \arccos(-1 + r)$ to $p_F = \arccos(-1 + r + O(\lambda r))$ and, more remarkably, produces an anomalous behavior in the 2-point Schwinger function due to the presence of the critical exponent η. Luttinger liquid behavior persists up to the critical point (corresponding to a strong coupling phase in the effective relativistic description); interestingly, the critical exponent becomes smaller the closer one is to the critical point. This is due to the fact that the effective coupling is $O(\lambda r)$ (and not $O(\lambda)$), so that the effective coupling divided by the Fermi velocity is $O(\sqrt{r})$ and thus small for small r.

At the critical point no anomalous exponent is present; the asymptotic behavior is qualitatively the same as in the non-interacting case, up to a finite wave function renormalization and the presence of α. Finally in the $r < 0$ again an insulating behavior is found, as the 2-point function has no singularities.

1.4. Grassmann Representation

To prove the above Theorem, it is convenient to write the Schwinger function in terms of *Grassmann integrals*. We introduce the propagator

$$g_{M,L,\beta}(\mathbf{x} - \mathbf{y}) = \frac{1}{\beta L} \sum_{\mathbf{k} \in \mathscr{D}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{\chi_0(\gamma^{-M}|k_0|)}{-ik_0 + \cos k + h} \tag{16}$$

where $\chi_0(t)$ is a smooth even compact support function with $\chi_0(t) = 1$ for $|t| \leq 1$, $\chi_0(t) > 0$ for $1 < t < \gamma$ and $\chi_0(t) = 0$ for $|t| \geq \gamma$, for some $1 < \gamma \leq 2$.

Let $\mathscr{D}_\beta$ be the subset of $\mathscr{D}$ contained in the support of $\chi_0(\gamma^{-M}|k_0|)$, that is $\mathscr{D}_\beta = \{\mathbf{k} \in \mathscr{D} \,|\, |k_0| < \gamma^{M+1}\}$. We consider the Grassmann algebra generated by the anticommuting Grassmannian variables $\{\psi_{\mathbf{k}}^{\pm}\}_{\mathbf{k} \in \mathscr{D}_\beta}$. On this algebra we define the Grassmann integration $\int \left[\prod_{\mathbf{k} \in \mathscr{D}_\beta} d\psi_{\mathbf{k}}^{-} d\psi_{\mathbf{k}}^{+}\right]$ as the linear operator defined

$$\int \left[\prod_{\mathbf{k} \in \mathscr{D}_\beta} d\psi_{\mathbf{k}}^{+} d\psi_{\mathbf{k}}^{-}\right] \prod_{\mathbf{k} \in \mathscr{D}_\beta} \psi_{\mathbf{k}}^{-} \psi_{\mathbf{k}}^{+} = 1 \tag{17}$$

while

$$\int \left[\prod_{\mathbf{k} \in \mathscr{D}_\beta} d\psi_{\mathbf{k}}^{+} d\psi_{\mathbf{k}}^{-}\right] Q(\psi^{-}, \psi^{+}) = 0 \tag{18}$$

if the monomial $Q(\psi^{-}, \psi^{+})$ does not contains all of the variables $\{\psi_{\mathbf{k}}^{\pm}\}_{\mathbf{k} \in \mathscr{D}_\beta}$. Finally this definition can be extended to the all algebra using the anticommuting properties of the Grassmannian variables and linearity.

Setting $a = \frac{\pi}{\gamma^{M+1}}$, $\Gamma_\beta = \left\{x_0 = m_0 a \,|\, 0 \leq m_0 \leq \left[\frac{\beta}{a}\right] - 1\right\}$, for $\mathbf{x} \in \Gamma_\beta \times \Lambda$ we define the *Grassmannian fields* as $\psi_{\mathbf{x}}^{\pm} = \frac{1}{\beta L} \sum_{\mathbf{k} \in \mathscr{D}_\beta} e^{\pm i\mathbf{k}\mathbf{x}} \psi_{\mathbf{k}}^{\pm}$, while the *Gaussian Grassmann measure* is defined as

$$P(d\psi) = \left[\prod_{\mathbf{k} \in \mathscr{D}_\beta} \beta L d\psi_{\kappa}^{-} d\psi_{\mathbf{k}}^{+} \hat{g}^{(\leq M)}(\mathbf{k})\right] \exp\left\{-\frac{1}{\beta L} \sum_{\mathbf{k} \in \mathscr{D}_\beta} (\hat{g}^{(\leq M)}(\mathbf{k}))^{-1} \psi_{\mathbf{k}}^{+} \psi_{\mathbf{k}}^{-}\right\}. \tag{19}$$

After extending the Grassmann algebra adding the new anticommuting Grassmann fields $\{\phi_{\mathbf{x}}^{\pm}\}_{\mathbf{x} \in \Gamma_\beta \times \Lambda}$, we introduce the generating functional $\mathscr{W}_M(\phi)$ defined in terms of the following Grassmann integral

$$e^{-\mathscr{W}_M(\phi)} = \int P(d\psi) e^{-\mathscr{V}(\psi) + (\psi, \phi)} \tag{20}$$

where, if $\int d\mathbf{x}$ is a short form for $\sum_{x \in \Lambda} a \sum_{x_0 \in \Gamma_\beta}$, we set

$$(\psi, \phi) = \int d\mathbf{x} [\psi_{\mathbf{x}}^{+} \phi_{\mathbf{x}}^{-} + \psi_{\mathbf{x}}^{-} \phi_{\mathbf{x}}^{+}] \tag{21}$$

$$\mathscr{V}(\psi) = \lambda \int d\mathbf{x} d\mathbf{y} v(\mathbf{x} - \mathbf{y}) \psi_{\mathbf{x}}^{+} \psi_{\mathbf{x}}^{-} \psi_{\mathbf{y}}^{+} \psi_{\mathbf{y}}^{-} + \bar{\nu} \int d\mathbf{x} \psi_{\mathbf{x}}^{+} \psi_{\mathbf{x}}^{-} \tag{22}$$

and $v(\mathbf{x} - \mathbf{y}) = \delta(x_0 - y_0) v(x - y)$; moreover,

$$\bar{\nu} = \lambda \hat{v}(0) \left[\frac{S_{0,L,\beta}(0^{+}, 0) - S_{0,L,\beta}(0^{-}, 0)}{2}\right]. \tag{23}$$

We call $\lim_{M\to\infty} g_{M,L,\beta}(\mathbf{x}) = g_{L,\beta}(\mathbf{x})$ and we observe that if $\mathbf{x} \neq (n\beta, 0)$

$$g_{L,\beta}(\mathbf{x}) = S_{0,L,\beta}(\mathbf{x}) \tag{24}$$

while for $\mathbf{x} = (n\beta, 0)$

$$g_{L,\beta}(\mathbf{x}) = \frac{S_{0,L,\beta}(n\beta^+, 0) + S_{0,L,\beta}(n\beta^-, 0)}{2} \tag{25}$$

and $S_{0,L,\beta}(n\beta, 0) = (-1)^n S_{0,L,\beta}(0^-, 0)$.

Finally we define

$$S^M_{L,\beta}(\mathbf{x} - \mathbf{y}) = \frac{\partial^2}{\partial\phi^+_{\mathbf{x}} \partial\phi^-_{\mathbf{y}}} \mathscr{W}_M(\phi)|_{\phi=0}. \tag{26}$$

The Grassmann integral (26) can be used to compute the thermodynamical properties of the model with Hamiltonian (1); a sketch of the proof of this well-known fact is in Appendix A.

1.5. Setting up the Multiscale Analysis

The analysis of the Grassmann integral (20) is done using a multiscale expansion. For definiteness, we take $|r| \leq 1/4$. The remaining range of r is covered by th e results in [4]. The starting point of the analysis is the following decomposition of the propagator

$$g_{M,L,\beta}(\mathbf{x}) = g^{(>0)}(\mathbf{x}) + g^{(\leq 0)}(\mathbf{x}) \tag{27}$$

where

$$g^{(\leq 0)}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_0(\gamma^{-M}|k_0|)\chi_{\leq 0}(\mathbf{k})}{-ik_0 + \cos k + h} \tag{28}$$

where $\int d\mathbf{k}$ stands for $\frac{1}{\beta L}\sum_{\mathbf{k}\in\mathscr{D}_\beta}$, $\chi_{\leq 0}(\mathbf{k}) = \chi_0\left(a_0^{-1}\sqrt{k_0^2 + (\cos k - 1 + r)^2}\right)$.

Finally $g^{(>0)}(\mathbf{x})$ is equal to (28) with $\chi_{\leq 0}(\mathbf{k})$ replaced by $(1 - \chi_{\leq 0}(\mathbf{k}))$. We chose $a_0 = \gamma^{-1}(1/2 - r)$ so that, in the support of $\chi_{\leq 0}(\mathbf{k})$ we have $|k| \leq \pi/6$. This assures that on the domain of $\chi_{\leq 0}$ we have

$$\frac{1}{2}|k| \leq |\sin(k)| \leq |k|. \tag{29}$$

Using the addition property of Grassmann integrations we can write

$$e^{-\mathscr{W}(\phi)} = \int P(d\psi^{(>0)})P(d\psi^{(\leq 0)})e^{-\mathscr{V}(\psi^{(>0)}+\psi^{(\leq 0)})+(\psi^{(>0)}+\psi^{(\leq 0)},\phi)}. \tag{30}$$

After integrating the field $\psi^{(>0)}$ one obtains

$$e^{-\mathscr{W}(\phi)} = e^{-\beta L F_0} \int P(d\psi^{(\leq 0)})e^{-\mathscr{V}^{(0)}(\psi^{\leq 0},\phi)}. \tag{31}$$

It is known, see for instance Lemma 2.2 of [8] for a proof, that $\mathscr{V}^{(0)}(\psi^{(\leq 0)}, \phi)$ is given by

$$\mathscr{V}^{(0)}(\psi, \phi) = \sum_{n+m\geq 1} \int d\underline{\mathbf{x}} \int d\underline{\mathbf{y}} \prod_{i=1}^n \psi^{\varepsilon_i}_{\mathbf{x}_i} \prod_{j=1}^m \phi^{\sigma_j}_{\mathbf{x}_j} W_{n,m}(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \tag{32}$$

where $\underline{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\underline{\mathbf{y}} = (\mathbf{y}_1, \ldots, \mathbf{y}_m)$ while $\prod_{i=1}^{n} \psi_{\mathbf{x}_i}^{\varepsilon_i} = 1$ if $n = 0$ and $\prod_{j=1}^{m} \phi_{\mathbf{y}_j}^{\sigma_j} = 1$ if $m = 0$; moreover, $W_{n,m}(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ are given by convergent power series in λ for λ small enough and they decay faster than any power in any coordinate difference. Finally, the limit $M \to \infty$ of $\mathscr{V}^{(0)}(\psi, \phi)$ exists and is reached uniformly in β, L.

Thus, we are left with the integration over $\psi^{(\leq 0)}$. The idea to to perform this integration is to decompose $\psi_{\mathbf{x}}^{(\leq 0)}$ as

$$\psi_{\mathbf{x}}^{(\leq 0)} = \sum_{h=0}^{-\infty} \psi_{\mathbf{x}}^{(h)}$$

where $\psi_{\mathbf{x}}^{(h)}$ depends only on the momenta $\mathbf{k}$ such that $-ik_0 + \cos k - 1 + r \simeq \gamma^h$. Using repeatedly the addition property for Grassmann integration this decomposition should allow us to integrate recursively over the $\psi^{(h)}$. The index h is called the *scale* of the field $\psi^{(h)}$. When $r > 0$, two different regimes naturally appear in the analysis, separated by an energy scale depending on r and defined as

$$h^* = \inf\{h \,|\, a_0 \gamma^{h+1} > |r|\}. \tag{33}$$

A different implementation of the renormalization procedure is done in each of these two regimes.

1. The first regime correspond to $h \geq h^*$; the momentum k is far from the two Fermi points $\pm p_F$ and the dispersion relation is approximatively quadratic: $\cos k - h \sim k^2$. In this regime, due to Pauli principle (or anticommutativity of Grassmann variables), the effective interactions are much smaller than one can expect according to power counting arguments, and the theory turns out to be superrenormalizable: all the interactions are irrelevant and their effect is to produce a finite renormalization of the parameters. In the insulating phase or at the critical point, only this regime is present.

2. In the second regime, for $h < h^*$, the momentum k is close to one of the Fermi points and the leading contribution to the dispersion relation is linear; $\cos k - h \sim (k \pm p_F)$. The local quartic terms are, therefore, marginal and the theory is renormalizable. As a consequence, while in the previous regime the wave function renormalization is finite, in this regime increases at each iteration step. The wave function renormalization is, therefore, extracted at each iteration step and is exactly compensated by the growth of the effective coupling. In addition the small divisors due to the vanishing Fermi velocity are compensated by the factors coming from the integration of the first regime, due to the different scaling dimensions.

Finally, if $r \leq 0$ only the first regime is present.

2. Renormalization Group Integration: The First Regime

We saw that after the ultraviolet integration we have

$$\mathrm{e}^{-\mathscr{W}(0)} = \mathrm{e}^{-\beta L F_0} \int P(\mathrm{d}\psi^{(\leq 0)}) \mathrm{e}^{-\mathscr{V}^{(0)}(\psi^{(\leq 0)})} \tag{34}$$

where $\mathscr{V}^{(0)}(\psi^{(\leq 0)}) = \mathscr{V}^{(0)}(\psi, 0)$ is the *effective potential* on scale 0 and can be written as

$$\mathscr{V}^{(0)}(\psi) = \sum_{n \geq 1} \int \mathrm{d}\underline{\mathbf{x}} \int \mathrm{d}\underline{\mathbf{y}} W_{2n}^{(0)}(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \prod_{i=1}^{n} \psi_{\mathbf{x}_i}^{+} \psi_{\mathbf{y}_i}^{-} = \sum_{n \geq 1} \mathscr{V}_{2n}^{(0)}(\psi). \tag{35}$$

A direct perturbative analysis suggest that to perform the integration (34) we need a *renormalized* multiscale integration procedure. In particular, the terms with $n = 1, 2$ are *relevant* and the terms with $n = 3$ are *marginal*. For this reason we introduce a *localization operator* acting on the effective potential as

$$\mathscr{V}^{(0)} = \mathscr{L}_1 \mathscr{V}^{(0)} + \mathscr{R}_1 \mathscr{V}^{(0)} \tag{36}$$

with $\mathscr{R}_1 = 1 - \mathscr{L}_1$ and $\mathscr{R}_1$ is defined in the following way;

1. $\mathscr{R}_1 \mathscr{V}_{2n}^{(0)} = \mathscr{V}_{2n}^{(0)}$ for $n \geq 4$;
2. for $n = 3, 2$

$$\mathscr{R}_1 \mathscr{V}_4^{(0)}(\psi) = \int \prod_{i=1}^{4} \mathrm{d}\mathbf{x}_i W_4^{(0)}(\underline{\mathbf{x}}) \psi_{\mathbf{x}_1}^{+} D_{\mathbf{x}_2, \mathbf{x}_1}^{+} \psi_{\mathbf{x}_3}^{-} D_{\mathbf{x}_4, \mathbf{x}_3}^{-} \tag{37}$$

$$\mathscr{R}_1 \mathscr{V}_6^{(0)}(\psi) = \int \prod_{i=1}^{6} \mathrm{d}\mathbf{x}_i W_6^{(0)}(\underline{\mathbf{x}}) \psi_{\mathbf{x}_1}^{+} D_{\mathbf{x}_2, \mathbf{x}_1}^{+} D_{\mathbf{x}_3, \mathbf{x}_1}^{+} \psi_{\mathbf{x}_4}^{-} D_{\mathbf{x}_5, \mathbf{x}_4}^{-} D_{\mathbf{x}_6, \mathbf{x}_4}^{-} \tag{38}$$

where

$$D_{\mathbf{x}_2, \mathbf{x}_1}^{\varepsilon} = \psi_{\mathbf{x}_2}^{\varepsilon} - \psi_{\mathbf{x}_1}^{\varepsilon} \tag{39}$$

3. For $n = 1$

$$\mathscr{R}_1 \mathscr{V}_2^{(0)}(\psi) = \int \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 W_2^{(0)}(\underline{\mathbf{x}}) \psi_{\mathbf{x}_1}^{+} H_{\mathbf{x}_1, \mathbf{x}_2}^{-} \tag{40}$$

where

$$H_{\mathbf{x}_1, \mathbf{x}_2}^{-} = \psi_{\mathbf{x}_2}^{-} - \psi_{\mathbf{x}_1}^{-} - (x_{0,1} - x_{0,2}) \partial_0 \psi_{\mathbf{x}_1}^{-} - (x_1 - x_2) \widetilde{\partial}_1 \psi_{\mathbf{x}_1}^{-}$$
$$- \frac{1}{2}\left(x_1 - x_2\right)^2 \widetilde{\Delta}_1 \psi \tag{41}$$

and

$$\widetilde{\partial}_1 \psi_{\mathbf{x}}^{-} = \frac{1}{2}(\psi_{\mathbf{x}+(0,1)}^{-} - \psi_{\mathbf{x}-(0,1)}^{-}) = \int \mathrm{d}\mathbf{k}\, i \sin k\, \mathrm{e}^{i\mathbf{k}\mathbf{x}} \hat{\psi}_{\mathbf{k}}^{-}$$

$$\widetilde{\Delta}_1 \psi_{\mathbf{x}}^{-} = \psi_{\mathbf{x}+(0,1)}^{-} - 2\psi_{\mathbf{x}}^{-} + \psi_{\mathbf{x}-(0,1)}^{-} = 2 \int \mathrm{d}\mathbf{k}(\cos k - 1) \mathrm{e}^{i\mathbf{k}\mathbf{x}} \hat{\psi}_{\mathbf{k}}^{-}.$$

As a consequence of the above definitions, calling

$$\hat{W}_2^{(0)}(\mathbf{k}) = \int \mathrm{d}\mathbf{x} \mathrm{e}^{i\mathbf{k}\mathbf{x}} W_2^{(0)}(\mathbf{x}) \tag{42}$$

we get

$$\mathscr{L}_1 \mathscr{V}^{(0)} = \hat{W}_2^{(0)}(0) \int \mathrm{d}\mathbf{x}\psi_{\mathbf{x}}^+ \psi_{\mathbf{x}}^- + \partial_0 \hat{W}_2^{(0)}(0) \int \mathrm{d}\mathbf{x}\psi_{\mathbf{x}}^+ \partial_0 \psi_{\mathbf{x}}^-$$
$$+ \frac{1}{2}\partial_1^2 \hat{W}_2^{(0)}(0) \int \mathrm{d}\mathbf{x}\psi_{\mathbf{x}}^+ \widetilde{\Delta}_1 \psi_{\mathbf{x}}^- \tag{43}$$

where we have used that

i. $g^{(0)}(k_0, k) = g^{(0)}(k_0, -k)$, so that we get

$$\partial_1 \hat{W}_2^{(0)}(0) = 0 \tag{44}$$

ii. There are no terms in $\mathscr{L}_1 \mathscr{V}^{(0)}$ with four or six fermionic fields, as

$$\psi_{\mathbf{x}_1}^\varepsilon D_{\mathbf{x}_2,\mathbf{x}_1}^\varepsilon = \psi_{\mathbf{x}_1}^\varepsilon \psi_{\mathbf{x}_2}^\varepsilon \tag{45}$$

and, therefore, $\mathscr{R}_1 \mathscr{V}_4^{(0)} = \mathscr{V}_4^{(0)}$ and $\mathscr{R}_1 \mathscr{V}_6^{(0)} = \mathscr{V}_6^{(0)}$. As a consequence (37)(38) just represent a different way to write the four and six field contribution to the effective potential. This representation will be useful in the following where we will exploit the dimensional gain due to the zero term $\mathbf{x}_2 - \mathbf{x}_1$ and the derivative in Eq.(39).

We will call $\mathscr{L}_1 \mathscr{V}^{(h)}$ the *relevant part of the effective potential*. Since it is quadratic in the fields, we can include it in the free integration finding

$$\mathrm{e}^{-\mathscr{W}(0)} = \mathrm{e}^{-\beta L(F_0 + e_0)} \int \widetilde{P}(\mathrm{d}\psi^{(\leq 0)}) \mathrm{e}^{-\mathscr{R}_1 \mathscr{V}^{(0)}(\psi^{(\leq 0)})} \tag{46}$$

where e_0 comes from the normalization of the new Grassmann integration and the propagator of $\widetilde{P}(\mathrm{d}\psi^{(\leq 0)})$ is now

$$\widetilde{g}^{(\leq 0)}(\mathbf{x}) = \int \mathrm{d}\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_{\leq 0}(\mathbf{k})}{D_{-1}(\mathbf{k})} \tag{47}$$

where $D_{-1}(\mathbf{k}) = -ik_0(1 + z_{-1}) + (1 + \alpha_{-1})(\cos k - 1) + r + \gamma^{-1}\mu_{-1}$ and

$$z_{-1} = z_0 + \chi_{\leq 0}(\mathbf{k})\partial_0 \hat{W}_2^{(0)}(0) \qquad \alpha_{-1} = \alpha_0 + \chi_{\leq 0}(\mathbf{k})\partial_1^2 \hat{W}_2^{(0)}(0)$$
$$\gamma^{-1}\mu_{-1} = \mu_0 + \chi_{\leq 0}(\mathbf{k})\hat{W}_2^{(0)}(0) \tag{48}$$

where $z_0 = \alpha_0 = \mu_0 = 0$ but we have added them in (48) for later reference. The definition of μ_{-1} in the last of (48) will be useful to simplify the power counting estimates in the following section.

We can now write

$$\widetilde{g}^{(\leq 0)}(\mathbf{x}) = g^{(\leq -1)}(\mathbf{x}) + \widetilde{g}^{(0)}(\mathbf{x}) \tag{49}$$

where

$$g^{(\leq -1)}(\mathbf{x}) = \int \mathrm{d}\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_{\leq -1}(\mathbf{k})}{D_{-1}(\mathbf{k})} \tag{50}$$

with

$$\chi_{< -1}(\mathbf{k}) = \chi_0\left(\gamma a_0^{-1}|D_{-1}(\mathbf{k})|\right).$$

Clearly

$$\widetilde{g}^{(0)}(\mathbf{x}) = \int \mathrm{d}\mathbf{k}\,e^{i\mathbf{k}\mathbf{x}}\frac{f_0(\mathbf{k})}{D_{-1}(\mathbf{k})} \tag{51}$$

where

$$f_0(\mathbf{k}) = \chi_{\leq 0}(\mathbf{k}) - \chi_{\leq -1}(\mathbf{k}).$$

Using again the addition property for Grassmann integrations we can rewrite (34) and perform the integration over $\psi^{(0)}$ as

$$e^{-\mathscr{W}(0)} = e^{-\beta L(F_0 + e_0)}\int P(\mathrm{d}\psi^{(\leq -1)})\int \widetilde{P}(\mathrm{d}\psi^{(0)})e^{-\mathscr{R}_1\mathscr{V}^{(0)}(\psi^{(\leq 0)})} \tag{52}$$

$$= e^{-\beta L F_{-1}}\int P(\mathrm{d}\psi^{(\leq -1)})e^{-\mathscr{V}^{(-1)}(\psi^{(\leq -1)})} \tag{53}$$

where $\widetilde{P}(\mathrm{d}\psi^{(0)})$ is the integration with propagator $\widetilde{g}^{(0)}(\mathbf{x})$, $P(\mathrm{d}\psi^{(\leq 1)})$ is the integration with propagator $g^{(\leq 1)}(\mathbf{x})$ and

$$e^{-\beta L\widetilde{e}_0 - \mathscr{V}^{(-1)}(\psi^{(\leq -1)})} = \int \widetilde{P}(\mathrm{d}\psi^{(0)})e^{-\mathscr{R}_1\mathscr{V}^{(0)}(\psi^{(\leq 0)})} \tag{54}$$

and $F_{-1} = F_0 + e_0 + \widetilde{e}_0$. The fact that this integration is well defined follows from the properties of the propagator $\widetilde{g}^{(0)}(\mathbf{x})$ that will be derived in Lemma 2.1 below.

We can now repeat the above procedure iteratively. At the h step (i.e., at scale h) we start with the integration

$$e^{-\mathscr{W}(0)} = e^{-\beta L F_h}\int P(\mathrm{d}\psi^{(\leq h)})e^{-\mathscr{V}^{(h)}(\psi^{(\leq h)})} \tag{55}$$

defined by the propagator

$$g^{(\leq h)}(\mathbf{x}) = \int \mathrm{d}\mathbf{k}\,e^{i\mathbf{k}\mathbf{x}}\frac{\chi_{\leq h}(\mathbf{k})}{D_h(\mathbf{k})} \tag{56}$$

with $D_h(\mathbf{k}) = -ik_0(1 + z_h) + (1 + \alpha_h)(\cos k - 1) + r + \gamma^h\mu_h$ and

$$\chi_{\leq h}(\mathbf{k}) = \chi_0\left(\gamma^{-h}a_0^{-1}|D_h(\mathbf{k})|\right). \tag{57}$$

The effective potential on scale h is given by

$$\mathscr{V}^{(h)}(\psi) = \sum_{n\geq 1}\int \mathrm{d}\underline{\mathbf{x}}\int \mathrm{d}\underline{\mathbf{y}}W_{2n}^{(h)}(\underline{\mathbf{x}},\underline{\mathbf{y}})\prod_{i=1}^{n}\psi_{\mathbf{x}_i}^+\psi_{\mathbf{y}_i}^- = \sum_{n\geq 1}\mathscr{V}_{2n}^{(h)}(\psi).$$

Again we can apply the operator $\mathscr{L}_1$ to $\mathscr{V}^{(h)}$ to get

$$\mathscr{V}^{(h)} = \mathscr{L}_1\mathscr{V}^{(h)} + \mathscr{R}_1\mathscr{V}^{(h)}. \tag{58}$$

where $\mathscr{R}$ is defined exactly as in the case of $\mathscr{V}^{(0)}$ and

$$\mathscr{L}_1\mathscr{V}^{(h)} = \hat{W}_2^{(h)}(0)\int \mathrm{d}\mathbf{x}\psi_{\mathbf{x}}^+\psi_{\mathbf{x}}^- + \partial_0\hat{W}_2^{(h)}(0)\int \mathrm{d}\mathbf{x}\psi_{\mathbf{x}}^+\partial_0\psi_{\mathbf{x}}^-$$
$$+ \frac{1}{2}\partial_1^2\hat{W}_2^{(h)}(0)\int \mathrm{d}\mathbf{x}\psi_{\mathbf{x}}^+\widetilde{\Delta}_1\psi_{\mathbf{x}}^-. \tag{59}$$

Moving the relevant part of the effective potential into the integration we get

$$\mathrm{e}^{-\mathscr{W}(0)} = \mathrm{e}^{-\beta L(F_h + e_h)} \int \widetilde{P}(\mathrm{d}\psi^{(\leq h)}) \mathrm{e}^{-\mathscr{R}\mathscr{V}^{(h)}(\psi^{(\leq h)})} \tag{60}$$

where the propagator of $\widetilde{P}(\mathrm{d}\psi^{(\leq h)})$ is

$$\widetilde{g}^{(\leq h)}(\mathbf{x}) = \int \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\mathbf{x}} \frac{\chi_{\leq h}(\mathbf{k})}{D_{h-1}(\mathbf{k})} \tag{61}$$

and the *running coupling constants* are defined recursively by

$$z_{h-1} = z_h + \chi_{\leq h}(\mathbf{k})\partial_0 \hat{W}_2^{(h)}(0) \qquad \alpha_{h-1} = \alpha_h + \chi_{\leq h}(\mathbf{k})\partial_1^2 \hat{W}_2^{(h)}(0)$$
$$\mu_{h-1} = \gamma\mu_h + \chi_{\leq h}(\mathbf{k})\gamma^{-h}\hat{W}_2^{(h)}(0). \tag{62}$$

Finally we can rewrite (46) as

$$\mathrm{e}^{-\mathscr{W}(0)} = \mathrm{e}^{-\beta L(F_h + e_h)} \int P(\mathrm{d}\psi^{(\leq h-1)}) \int \widetilde{P}(\mathrm{d}\psi^{(h)}) \mathrm{e}^{-\mathscr{R}_1\mathscr{V}^{(h)}(\psi^{(\leq h)})} \tag{63}$$

where $\widetilde{P}(\mathrm{d}\psi^{(h)})$ has now propagator

$$\widetilde{g}^{(h)}(\mathbf{x}) = \int \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\mathbf{x}} \frac{f_h(\mathbf{k})}{D_{h-1}(\mathbf{k})}$$
$$= \int \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\mathbf{x}} \hat{g}^{(h)}(\mathbf{k}) \tag{64}$$

and $f_h(\mathbf{k}) = \chi_{\leq h}(\mathbf{k}) - \chi_{\leq h-1}(\mathbf{k})$; one can perform the integration over $\psi^{(h)}$

$$\mathrm{e}^{-\beta L\bar{e}_h - \mathscr{V}^{h-1}} = \int \widetilde{P}(\mathrm{d}\psi^{(h)}) \mathrm{e}^{-\mathscr{R}_1\mathscr{V}^{(h)}(\psi^{(\leq h)})} \tag{65}$$

obtaining an expression identical to (55) with $h - 1$ replacing h, so that the procedure can be iterated.

To show that the above procedure is well defined we need to study the propagator $\widetilde{g}^{(\leq h)}(\mathbf{x})$. We first have to distinguish two range of scales. The construction of the theory for $r > 0$, is based on the fact that the behavior of the propagator changes significantly when one reaches the scales $h \simeq h^*$ defined in (33). To understand this phenomenon, let us, for simplicity sake, neglect the presence of the running constant in the function $\chi_{\leq h}$. We will see in Lemma 2.1 and Lemma 3.1 below that the presence of α_h, z_h and μ_h does not change the picture. In this situation, it is easy to see that if $h > h^*$ then the domain of $f_h(\mathbf{k})$ is a ring of width γ^h that goes around both Fermi points $(0, \pm p_F)$. At this momentum scale the propagator does not distinguish between p_F and $-p_F$. On the other hand, when $h < h^*$ we have

$$k_0^2 + (\cos k - 1 + r)^2 > a_0^2 \gamma^{2h+1} \tag{66}$$

in an open neighbor of the k_0 axis. This means that the domain of $f_h(\mathbf{k})$ splits in two rings, one around p_F and the other around $-p_F$. In this situation it is convenient to write the propagator as a sum of two *quasi-particle* propagators, each of which depends only on the momenta close to one of the Fermi points.

Here, we need precise estimates on $\widetilde{g}^{(h)}$ for $h \geq h^*$ as reported in the following Lemma. The case $h < h^*$ will be studied in section 3. The prove of this Lemma is reported in Appendix B.

Lemma 2.1. *Assume that there exists a constant $K > 0$ such that*

$$|z_h|, |\alpha_h|, |\mu_h| < K|\lambda| \tag{67}$$

for $h \geq h^*$. Then for $|x_0| \leq \beta/2$, every N and λ small enough we have

$$\left| \partial_0^{n_0} \widetilde{\partial}_1^{n_1} \widetilde{g}^{(h)}(\mathbf{x}) \right| \leq C_N \frac{\gamma^{\frac{h}{2}}}{1 + [\gamma^h |x_0| + \gamma^{\frac{h}{2}} |x|]^N} \gamma^{h(n_0 + n_1/2)} \tag{68}$$

with C_N independent from K.

2.1. Tree Expansion for the Effective Potentials.

The effective potential $\mathscr{V}^{(h)}(\psi^{(\leq h)})$ can be written in terms of a *tree expansion*, see [10,11], defined as follows.

1. On the plane, we draw the vertical lines at horizontal position given by the integers from h to 1, see Fig. 1. We select one point on the line at h (*the root*) and one point on the line at $h + 1$ (the first vertex v_0). On the line at k, with $h + 1 < k \leq 1$, we select $m_k > 0$ points (the *vertices at scale k*). We call M_k the set of vertices at scale k. To each vertex v in M_k we associate exactly one vertex v' in M_{k-1} and we draw a line between these two vertices. The vertex v' is called the *predecessor* of v. Finally we require that if v and w are in M_k with v below w then v' is below or equal to w'. The final results of this procedure is clearly a tree with root r.

2. Given a vertex v on scale k, let s_v be the number of vertices on scale $k + 1$ linked to v. If $s_v = 0$ we say that v is a *endpoint*. The number n of endpoint is called the *order* of the tree. If $s_v = 1$ we say that v is a *trivial* vertex. Finally if $s_v > 1$ we say that v is a *branching point* or *non-trivial* vertex. The tree structure induce a natural ordering (denoted
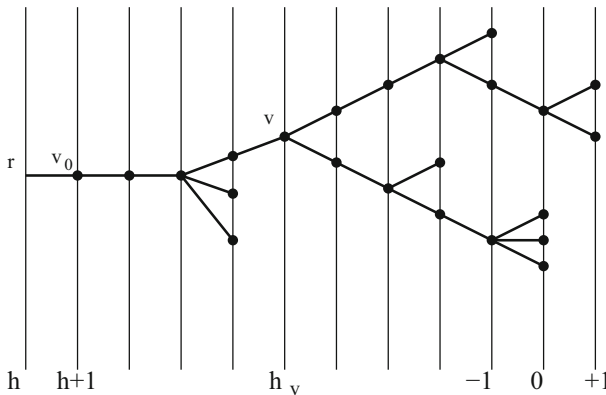

FIGURE 1. A tree $\tau \in \mathscr{T}_{h,n}$ with its scale labels

by $<$) on the vertex such that if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$. We call $\mathscr{T}_{h,n}$ the set of all tree constructed in this way.

3. Given a vertex v of $\tau \in \mathscr{T}_{h,n}$ that is not an endpoint, we can consider the subtrees of τ with root v, which correspond to the connected components of the restriction of τ to the vertices $w > v$. If a subtree with root v contains only v and an endpoint on scale $h_v + 1$, we will call it a *trivial subtree*.

4. With each endpoint v we associate one of the monomials contributing to $\mathscr{R}_1 \mathscr{V}^{(0)}(\psi^{(\leq h_v - 1)})$ and a set $\mathbf{x}_v$ of space-time points.

5. We introduce a *field label f* to distinguish the field variables appearing in the terms associated with the endpoints described in item (4); the set of field labels associated with the endpoint v will be called I_v, $\mathbf{x}(f)$, $\varepsilon(f)$ will be the position and type of the field variable f. Observe that $|I_v|$ is the order of the monomial contributing to $\mathscr{V}^{(0)}(\psi^{(\leq h_v - 1)})$ and associated to v. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; finally we will call the set of point $\mathbf{x}(f)$ for $f \in I_v$ the *cluster* associated to v.

Given $\mathscr{U}_i(\psi^{(h)})$ for $i = 1, \dots, n$ we define the *truncated expectation* on scale h as

$$
\mathscr{E}_h^T\big[\mathscr{U}_1(\psi^{(h)}); \dots; \mathscr{U}_n(\psi^{(h)})\big]
$$
$$
= \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \log \int \widetilde{P}\left(\mathrm{d}\psi^{(h)}\right) e^{\lambda_1 \mathscr{U}_1(\psi^{(h)}) + \cdots + \lambda_n \mathscr{U}_n(\psi^{(h)})}\bigg|_{\lambda_1 = \cdots = \lambda_N = 0}.
$$
(69)

In terms of above trees, the effective potential $\mathscr{V}^{(h)}$, $h \leq -1$, can be written as

$$
\mathscr{V}^{(h)}(\psi^{(\leq h)}) + \beta L \bar{e}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathscr{T}_{h,n}} \mathscr{V}^{(h)}(\tau, \psi^{(\leq h)}),
$$
(70)

where, if v_0 is the first vertex of τ and $\tau_1, \dots, \tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0, $\mathscr{V}^{(h)}(\tau, \psi^{(\leq h)})$ is defined inductively as follows:

i if $s > 1$, then

$$
\mathscr{V}^{(h)}(\tau, \psi^{(\leq h)})
$$
$$
= \frac{(-1)^{s+1}}{s!} \mathscr{E}_{h+1}^T\Big[\bar{\mathscr{V}}^{(h+1)}(\tau_1, \psi^{(\leq h+1)}); \dots; \bar{\mathscr{V}}^{(h+1)}(\tau_s, \psi^{(\leq h+1)})\Big],
$$
(71)

where $\bar{\mathscr{V}}^{(h+1)}(\tau_i, \psi^{(\leq h+1)})$ is equal to $\mathscr{R}_1 \mathscr{V}^{(h+1)}(\tau_i, \psi^{(\leq h+1)})$ if the subtree τ_i contains more than one endpoint, or if it contains one endpoint but it is not a trivial subtree; it is equal to $\mathscr{R}_1 \mathscr{V}^{(0)}(\psi^{(\leq h+1)})$ if τ_i is a trivial subtree;

ii if $s = 1$ and τ_1 is not a trivial subtree, then $\mathscr{V}^{(h)}(\tau, \psi^{(\leq h)})$ is equal to

$$
\mathscr{E}_{h+1}^T\Big[\mathscr{R}_1 \mathscr{V}^{(h+1)}(\tau_1, \psi^{(\leq h+1)})\Big].
$$

Using its inductive definition, the right-hand side of (70) can be further expanded, and to describe the resulting expansion we need some more definitions.

We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \ldots, v_{s_v}$ are the $s_v \geq 1$ vertices immediately following it, then $P_v \subseteq \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. If v is not an endpoint, we shall denote by Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The union $\mathscr{I}_v$ of the subsets $P_{v_i} \backslash Q_{v_i}$ is, by definition, the set of the *internal fields* of v, and is non-empty if $s_v > 1$. Given $\tau \in \mathscr{T}_{h,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with all the constraints. We shall denote $\mathscr{P}_\tau$ the family of all these choices and $\mathbf{P}$ the elements of $\mathscr{P}_\tau$.

With these definitions, we can rewrite $\mathscr{V}^{(h)}(\tau, \psi^{(\leq h)})$ in the r.h.s. of (70) as:

$$\mathscr{V}^{(h)}(\tau, \psi^{(\leq h)}) = \sum_{\mathbf{P} \in \mathscr{P}_\tau} \mathscr{V}^{(h)}(\tau, \mathbf{P}),$$

$$\mathscr{V}^{(h)}(\tau, \mathbf{P}) = \int d\mathbf{x}_{v_0} \widetilde{\psi}^{(\leq h)}(P_{v_0}) K_{\tau, \mathbf{P}}^{(h+1)}(\mathbf{x}_{v_0}), \tag{72}$$

where

$$\widetilde{\psi}^{(\leq h)}(P_v) = \prod_{f \in P_v} \psi_{\mathbf{x}(f)}^{(\leq h)\varepsilon(f)} \tag{73}$$

and $K_{\tau, \mathbf{P}}^{(h+1)}(\mathbf{x}_{v_0})$ is defined inductively by the equation, valid for any $v \in \tau$ which is not an endpoint,

$$K_{\tau, \mathbf{P}}^{(h_v)}(\mathbf{x}_v) = \frac{1}{s_v!} \prod_{i=1}^{s_v} \left[K_{v_i}^{(h_v+1)}(\mathbf{x}_{v_i}) \right] \; \mathscr{E}_{h_v}^T$$

$$\times [\widetilde{\psi}^{(h_v)}(P_{v_1} \backslash Q_{v_1}), \ldots, \widetilde{\psi}^{(h_v)}(P_{v_{s_v}} \backslash Q_{v_{s_v}})], \tag{74}$$

where $\widetilde{\psi}^{(h_v)}(P_{v_i} \backslash Q_{v_i})$ has a definition similar to (73). Moreover, if v_i is an endpoint $K_{v_i}^{(h_v+1)}(\mathbf{x}_{v_i})$ is equal to one of the kernels of the monomials contributing to $\mathscr{R}_1 \mathscr{V}^{(0)}(\psi^{(\leq h_v)})$; if v_i is not an endpoint, $K_{v_i}^{(h_v+1)} = K_{\tau_i, \mathbf{P}_i}^{(h_v+1)}$, where $\mathbf{P}_i = \{P_w, w \in \tau_i\}$.

The final form of our expansion is not yet given by (70)–(74). We can further decompose $\mathscr{V}^{(h)}(\tau, \mathbf{P})$, using the following representation of the truncated expectation in the r.h.s. of (74). Let us put $s = s_v$, $P_i \equiv P_{v_i} \backslash Q_{v_i}$; moreover, we order in an arbitrary way the sets $P_i^\pm \equiv \{f \in P_i, \varepsilon(f) = \pm\}$, we call $f_{ij}^\pm$ their elements and we define $\mathbf{x}^{(i)} = \cup_{f \in P_i^-} \mathbf{x}(f)$, $\mathbf{y}^{(i)} = \cup_{f \in P_i^+} \mathbf{x}(f)$, $\mathbf{x}_{ij} = \mathbf{x}(f_{ij}^-)$, $\mathbf{y}_{ij} = \mathbf{x}(f_{ij}^+)$. Note that $\sum_{i=1}^s |P_i^-| = \sum_{i=1}^s |P_i^+| \equiv n$, otherwise the truncated expectation vanishes.

Then, we use the *Brydges–Battle–Federbush* [12–14] formula saying that, up to a sign, if $s > 1$,

$$\mathscr{E}_h^T(\widetilde{\psi}^{(h)}(P_1), \ldots, \widetilde{\psi}^{(h)}(P_s)) = \sum_T \prod_{l \in T} g^{(h)}(\mathbf{x}_l - \mathbf{y}_l) \int dP_T(\mathbf{t}) \det G^{h,T}(\mathbf{t}),$$
(75)

where T is a set of lines forming an *anchored tree graph* between the clusters associated with v_i that is T is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover, $\mathbf{t} = \{t_{ii'} \in [0,1], 1 \leq i, i' \leq s\}$, $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{ii'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^s$ of unit norm. Finally $G^{h,T}(\mathbf{t})$ is a $(n - s + 1) \times (n - s + 1)$ matrix, whose elements are given by

$$G_{ij,i'j'}^{h,T} = t_{ii'} g^{(h)}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'}),$$
(76)

with $(f_{ij}^-, f_{i'j'}^+)$ not belonging to T. In the following we shall use (75) even for $s = 1$, when T is empty, by interpreting the r.h.s. as equal to 1, if $|P_1| = 0$, otherwise as equal to $\det G^h = \mathscr{E}_h^T(\widetilde{\psi}^{(h)}(P_1))$. It is crucial to note that $G^{h,T}$ is a Gram matrix, i.e., the matrix elements in (76) can be written in terms of scalar products:

$$t_{ii'} g^{(h)}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'}) = \left(\mathbf{u}_i \otimes A(\mathbf{x}_{ij} - \cdot), \ \mathbf{u}_{i'} \otimes B(\mathbf{x}_{i'j'} - \cdot) \right) \equiv (\mathbf{f}_\alpha, \mathbf{g}_\beta), \quad (77)$$

where

$$A(\mathbf{x}) = \int d\mathbf{k} e^{-i\mathbf{k}\mathbf{x}} \sqrt{f_h(\mathbf{k})} \frac{\sqrt{|D_h(\mathbf{k})|}}{D_h(\mathbf{k})},$$
(78)

$$B(\mathbf{x}) = \int d\mathbf{k} e^{-i\mathbf{k}\mathbf{x}} \sqrt{f_h(\mathbf{k})} \frac{1}{\sqrt{|D_h(\mathbf{k})|}}$$

with $D_h(\mathbf{k}) = -ik_0(1 + z_h) + (1 + \alpha_h) \cos k - 1 + r + \gamma^h \mu_h$. The symbol $(\cdot, \cdot)$ denotes the inner product, i.e.,

$$\left(\mathbf{u}_i \otimes A(\mathbf{x} - \cdot), \mathbf{u}_{i'} \otimes B(\mathbf{x}' - \cdot) \right) = (\mathbf{u}_i \cdot \mathbf{u}_{i'}) \cdot \int d\mathbf{z} A^*(\mathbf{x} - \mathbf{z}) B(\mathbf{x}' - \mathbf{z}), \quad (79)$$

and the vectors $\mathbf{f}_\alpha, \mathbf{g}_\beta$ with $\alpha, \beta = 1, \ldots, n - s + 1$ are implicitly defined by (77). The usefulness of the representation (77) is that, by the Gram–Hadamard inequality, $|\det(\mathbf{f}_\alpha, \mathbf{g}_\beta)| \leq \prod_\alpha ||\mathbf{f}_\alpha|| \, ||\mathbf{g}_\alpha||$. In our case, $||\mathbf{f}_\alpha||, ||\mathbf{g}_\alpha|| \leq C\gamma^{h/4}$ as it easily follows along the line of the proof of Lemma 2.1. Therefore, $||\mathbf{f}_\alpha|| \, ||\mathbf{g}_\alpha|| \leq C\gamma^{\frac{h}{2}}$, uniformly in α, so that the Gram determinant can be bounded by $C^{n-s+1} \gamma^{\frac{h}{2}(n-s+1)}$.

If we apply the expansion (75) in each vertex of τ different from the endpoints, we get an expression of the form

$$\mathscr{V}^{(h)}(\tau, \mathbf{P}) = \sum_{T \in \mathbf{T}} \int d\mathbf{x}_{v_0} \widetilde{\psi}^{(\leq h)}(P_{v_0}) W_{\tau, \mathbf{P}, T}^{(h)}(\mathbf{x}_{v_0}) = \sum_{T \in \mathbf{T}} \mathscr{V}^{(h)}(\tau, \mathbf{P}, T), \quad (80)$$

where $\mathbf{T}$ is a special family of graphs on the set of points $\mathbf{x}_{v_0}$, obtained by putting together an anchored tree graph T_v for each non-trivial vertex v. Note that any graph $T \in \mathbf{T}$ becomes a tree graph on $\mathbf{x}_{v_0}$, if one identifies all the points in the sets $\mathbf{x}_v$, with v an endpoint.

2.2. Analyticity of the Effective Potentials.

Our next goal is the proof of the following result.

Lemma 2.2. *There exists a constants $\lambda_0 > 0$, independent of β, L and r, such that the kernels $W_l^{(h)}$ in the domain $|\lambda| \leq \lambda_0$, are analytic function of λ and satisfy for $h \geq h^*$*

$$\frac{1}{\beta L} \int d\mathbf{x}_1 \cdots d\mathbf{x}_l |W_l^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_l)| \leq \gamma^{h\left(\frac{3}{2} - \frac{l}{4}\right)} \gamma^{\vartheta h} \left(C|\lambda|\right)^{\max(1, l-1)} \quad (81)$$

with $\vartheta = \frac{1}{4}$.

Proof. The proof is done by induction on h. We assume that for $k \geq h + 1$ (81) holds together with

$$\int d\mathbf{x} \left(|x_0| + |x_1|^2\right) |W_2^{(k)}(\mathbf{x})| \leq C|\lambda| \gamma^{\vartheta k} \quad (82)$$

and

$$\int d\mathbf{x} |W_2^{(k)}(\mathbf{x})| \leq C|\lambda||r| \gamma^{\vartheta k}. \quad (83)$$

The validity of (82) and (83) implies (67).

We now prove that the validity of (81), (82) and (83). Using the tree expansion described above and, in particular, (70), (72), (80), we find that the l.h.s. of (81) can be bounded above by

$$\sum_{n \geq 1} \sum_{\tau \in \mathscr{T}_{h,n}} \sum_{\substack{\mathbf{P} \in \mathscr{P}_\tau \\ |P_{v_0}| = l}} \sum_{T \in \mathbf{T}} C^n \left[\prod_{i=1}^{n} C^{p_i} |\lambda|^{\frac{p_i}{2} - 1}\right]$$

$$\times \left[\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{h_v\left(\sum_{i=1}^{s_v} \frac{|P_{v_i}|}{4} - \frac{|P_v|}{4} - \frac{3}{2}(s_v - 1)\right)}\right]$$

$$\times \left[\prod_{v \text{ not e.p.}} \gamma^{-(h_v - h_{v'})z_1(P_v)}\right] \left[\prod_{v \text{ e.p.}, |I_v| = 4,6} \gamma^{h_{v'} \frac{|I_v| - 2}{2}}\right] \left[\prod_{v \text{ e.p.}, |I_v| = 2} \gamma^{\frac{3h_{v'}}{2}}\right]$$

$$\tag{84}$$

where $z_1(P_v) = 2$ for $|P_v| = 6$, $z_1(P_v) = 1$ for $|P_v| = 4$ and $z_1(P_v) = \frac{3}{2}$ for $|P_v| = 2$. Note the role of the $\mathscr{R}_1$ operation in the above bound; if we neglect $\mathscr{R}_1$ we can get a similar bound where the second line of Eq. (84) is simply replaced by 1. Its proofs is an immediate consequence of the Gram–Hadamard inequality

$$|\det G^{h_v, T_v}(\mathbf{t}_v)| \leq c^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v| - 2(s_v - 1)} \cdot \gamma^{\frac{h_v}{2}\left(\sum_{i=1}^{s_v} \frac{|P_{v_i}|}{2} - \frac{|P_v|}{2} - (s_v - 1)\right)} \quad (85)$$

and of the decay properties of $g^{(h)}(\mathbf{x})$, implying

$$\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \int \prod_{l \in T_v} d(\mathbf{x}_l - \mathbf{y}_l) \, ||g^{(h_v)}(\mathbf{x}_l - \mathbf{y}_l)|| \leq c^n \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{-h_v(s_v - 1)}.$$

$$\tag{86}$$

If we take into account the subtraction to the 2 field terms and rewriting of the 4 and 6 fields terms involved in the $\mathscr{R}_1$ operation we obtain the extra factor

$$\left[\prod_{v \text{ not e.p.}} \gamma^{-(h_v - h_{v'})z_1(P_v)} \right] \left[\prod_{v \text{ e.p.}, |I_v|=4,6} \gamma^{h_{v'} \frac{|I_v|-2}{2}} \right] \left[\prod_{v \text{ e.p.}, |I_v|=2} \gamma^{\frac{3h_{v'}}{2}} \right]$$

which is produced by the extra zeros and derivatives in the fields $D_{\mathbf{x}_i, \mathbf{x}_j}$ [when written as in the last of (39)] and $H_{\mathbf{x}_1, \mathbf{x}_2}$; each time or space derivative produce a gain $\gamma^{h_{v'}}$ or $\gamma^{h_{v'}/2}$, respectively, while the zeros can be associated to the propagators in the anchored tree T (for vertices that are not end points) or to the kernels in $\mathscr{V}^{(0)}$ (for the end points) producing a loss bounded by γ^{-h_v} or $\gamma^{-h_v/2}$. While the origin of such factors can be easily understood by the above dimensional considerations, some care has to be taken to obtain such gains, related to the presence of the interpolated points and to avoid "bad" extra factorials; we refer for instance to section 3 of [4] where a similar bound in an analogous case is derived with all details.

Once the bound (84) is obtained, we have to see if we can sum over the scales and the trees. Let us define $n(v) = \sum_{i:v_i^* > v} 1$ as the number of endpoints following v on τ. Recalling that $|I_v|$ is the number of field labels associated to the endpoints following v on τ and using that

$$\sum_{v \text{ not e.p.}} \left[\sum_{i=1}^{s_v} |P_{v_i}| - |P_v| \right] = |I_{v_0}| - |P_{v_0}|,$$

$$\sum_{v \text{ not e.p.}} (s_v - 1) = n - 1, \tag{87}$$

$$\sum_{v \text{ not e.p.}} (h_v - h) \left[\sum_{i=1}^{s_v} |P_{v_i}| - |P_v| \right] = \sum_{v \text{ not e.p.}} (h_v - h_{v'})(|I_v| - |P_v|),$$

$$\sum_{v \text{ not e.p.}} (h_v - h)(s_v - 1) = \sum_{v \text{ not e.p.}} (h_v - h_{v'})(n(v) - 1),$$

we find that (84) can be bounded above by

$$\sum_{n \geq 1} \sum_{\tau \in \mathscr{T}_{h,n}} \sum_{\substack{\mathbf{P} \in \mathscr{P}_\tau \\ |P_{v_0}|=2l}} \sum_{T \in \mathbf{T}} C^n \gamma^{h\left(\frac{3}{2} - \frac{1}{4}|P_{v_0}| + \frac{1}{4}|I_{v_0}| - \frac{3}{2}n \right)} \left[\prod_{i=1}^{n} C^{p_i} |\lambda|^{\frac{p_i}{2}-1} \right]$$

$$\left[\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{(h_v - h_{v'})\left(\frac{3}{2} - \frac{|P_v|}{4} + \frac{|I_v|}{4} - \frac{3}{2}n(v) + z_1(P_v) \right)} \right]$$

$$\left[\prod_{v \text{ e.p.}, |I_v|=4,6} \gamma^{h_{v'} \frac{|I_v|-2}{2}} \right] \left[\prod_{v \text{ e.p.}, |I_v|=2} \gamma^{\frac{3h_{v'}}{2}} \right] \tag{88}$$

Using the identities

$$\gamma^{hn} \prod_{v \text{ not e.p.}} \gamma^{(h_v - h_{v'})n(v)} = \prod_{v \text{ e.p.}} \gamma^{h_{v'}},$$

$$\gamma^{h|I_{v_0}|} \prod_{v \text{ not e.p.}} \gamma^{(h_v - h_{v'})|I_v|} = \prod_{v \text{ e.p.}} \gamma^{h_{v'}|I_v|}, \tag{89}$$

we obtain

$$\frac{1}{\beta L} \int d\mathbf{x}_1 \cdots d\mathbf{x}_l |W_l^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_l)|$$

$$\leq \sum_{n \geq 1} \sum_{\tau \in \mathscr{T}_{h,n}} \sum_{\substack{\mathbf{P} \in \mathscr{P}_\tau \\ |P_{v_0}| = 2l}} \sum_{T \in \mathbf{T}} C^n \gamma^{h\left(\frac{3}{2} - \frac{l}{4}\right)} \left[\prod_{i=1}^{n} C^{p_i} |\lambda|^{\frac{p_i}{2} - 1} \right]$$

$$\times \left[\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{-(h_v - h_{v'})\left(\frac{|P_v|}{4} - \frac{3}{2} + z_1(P_v)\right)} \right]$$

$$\times \left[\prod_{v \text{ e.p.},|I_v|>6} \gamma^{h_{v'}\left(\frac{|I_v|}{4} - \frac{3}{2}\right)} \right] \left[\prod_{v \text{ e.p.},|I_v|=2} \gamma^{\frac{h_{v'}}{2}} \right] \left[\prod_{v \text{ e.p.},|I_v|=4,6} \gamma^{h_{v'}\frac{3|I_v|-10}{4}} \right]. \tag{90}$$

Note that,

$$\left[\prod_{v \text{ e.p.},|I_v|>6} \gamma^{h_{v'}\left(\frac{|I_v|}{4} - \frac{3}{2}\right)} \right] \left[\prod_{v \text{ e.p.},|I_v|=2} \gamma^{\frac{h_{v'}}{2}} \right] \left[\prod_{v \text{ e.p.},|I_v|=4,6} \gamma^{h_{v'}\frac{3|I_v|-10}{4}} \right] \leq \gamma^{\frac{\bar{h}}{2}}, \tag{91}$$

with $\bar{h}$ the highest scale label of the tree. Since

$$\frac{|P_v|}{4} - \frac{3}{2} + z_1(P_v) \geq \frac{1}{2} \tag{92}$$

we see that

$$\left[\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{-(h_v - h_{v'})\left(\frac{|P_v|}{4} - \frac{3}{2} + z_1(P_v)\right)} \right] \gamma^{\frac{\bar{h}}{2}}$$

$$\leq \left[\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{-(h_v - h_{v'})\eta\left(\frac{|P_v|}{4} - \frac{3}{2} + z_1(P_v)\right)} \right] \gamma^{h\frac{(1-\eta)}{2}}. \tag{93}$$

for any $0 < \eta < 1$. On the other hand we have that

$$\frac{|P_v|}{4} - \frac{3}{2} + z_1(P_v) \geq \frac{|P_v|}{16} \tag{94}$$

so that, using also Eq. (92), we get

$$\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{-(h_v - h_{v'})\eta\left(\frac{|P_v|}{4} - \frac{3}{2} + z_1(P_v)\right)}$$

$$\leq \left[\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{-\frac{\eta}{4}(h_v - h_{v'})} \right] \left[\prod_{v \text{ not e.p.}} \gamma^{-\frac{\eta}{32}|P_v|} \right]. \tag{95}$$

Collecting the above estimates and using that the number of terms in $\sum_{T \in \mathbf{T}}$ is bounded by $C^n \prod_{v \text{ not e.p.}} s_v!$ we obtain

$$\frac{1}{\beta L} \int d\mathbf{x}_1 \cdots d\mathbf{x}_l |W_l^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_l)|$$

$$\leq \gamma^{h\left(\frac{3}{2} - \frac{l}{4}\right)} \gamma^{\frac{1-\eta}{2}h} \sum_{n \geq 1} C^n \sum_{\tau \in \mathscr{T}_{h,n}} \left[\prod_{i=1}^n C^{p_i} |\lambda|^{\frac{p_i}{2} - 1} \right]$$

$$\times \left[\prod_{v \text{ not e.p.}} \gamma^{-(h_v - h_{v'})\frac{\eta}{4}} \right] \sum_{\substack{\mathbf{P} \in \mathscr{P}_\tau \\ |P_{v_0}| = 2l}} \left[\prod_{v \text{ not e.p.}} \gamma^{-\frac{|P_v|}{64}} \right]. \tag{96}$$

Remark. Equation (96) says that a gain $\gamma^{\frac{\bar{h}}{2}}$ at the scale of the endpoint, see (91), implies a gain $\gamma^{h\frac{1-\eta}{2}}$ at the root scale, as consequence of the fact that the renormalized scaling dimension of all vertices of the trees is strictly positive and $\geq \frac{1}{2}$; this property, which will be extensively used below, is called *short memory property*.

The sum over $\mathbf{P}$ can be bounded using the following combinatorial inequality: let $\{p_v, v \in \tau\}$, with $\tau \in \mathscr{T}_{h,n}$, be a set of integers such that $p_v \leq \sum_{i=1}^{s_v} p_{v_i}$ for all $v \in \tau$ which are not endpoints; then, if $\alpha > 0$,

$$\prod_{v \text{ not e.p.}} \sum_{p_v} \gamma^{-\alpha p_v} \leq C_\alpha^n.$$

This implies that

$$\sum_{\substack{\mathbf{P} \in \mathscr{P}_\tau \\ |P_{v_0}| = 2l}} \left[\prod_{v \text{ not e.p.}} \gamma^{-|P_v|\frac{\eta}{32}} \right] \prod_{i=1}^n C^{p_i} |\lambda|^{\frac{p_i}{2} - 1} \leq C^n |\lambda|^n.$$

Finally

$$\sum_{\tau \in \mathscr{T}_{h,n}} \prod_{v \text{ not e.p.}} \gamma^{-(h_v - h_{v'})\frac{\eta}{4}} \leq C^n,$$

as it follows from the fact that the number of non-trivial vertices in τ is smaller than $n - 1$ and that the number of trees in $\mathscr{T}_{h,n}$ is bounded by const^n. Altogether we obtain

$$\frac{1}{\beta L} \int d\mathbf{x}_1 \cdots d\mathbf{x}_l |W_l^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_l)| \leq \gamma^{h\left(\frac{3}{2} - \frac{l}{4}\right)} \gamma^{\vartheta h} \sum_{n \geq 1} C^n |\lambda|^n, \tag{97}$$

where we have set $\vartheta = (1 - \eta)/2$. Moreover, we choose $\eta = \frac{1}{2}$ so that $\vartheta = \frac{1}{4}$. Once convergence is established, the limit $L, \beta \to \infty$ is a straightforward consequence, see for instance section 2 of [8].

To complete the proof we need to show the validity of the inductive assumption (82)(83). It is clearly true for $h = 1$; moreover, by the bound (97) we get (82). We have finally to prove (83). We can write $g^{(h)}(\mathbf{x}) = g^{(h)}|_{r=0}(\mathbf{x}) + r^{(h)}(\mathbf{x})$ where $g^{(h)}|_{r=0}$ is the single scale propagator of the $r = 0$ case and $r^{(h)}$ satisfies

$$\left| \partial_0^{n_0} \widetilde{\partial}_1^{n_1} \widetilde{g}^{(h)}(\mathbf{x}) \right| \leq C_N \frac{|r|\gamma^{-\frac{h}{2}}}{1 + \left[\gamma^h |x_0| + \gamma^{\frac{h}{2}}|x| \right]^N} \gamma^{h(n_0 + n_1/2)}$$

that is the same bound (68) with an extra $\frac{|r|}{\gamma^h}$. We can, therefore, write

$$\hat{W}_2^{(h)}(0) = \hat{W}_{2,a}^{(h)}(0) + \hat{W}_{2,b}^{(h)}(0) \tag{98}$$

where $\hat{W}_{2,a}^{(h)}(0)$ is the effective potential of the $r = 0$ case. We will show below that $\sum_{h=-\infty}^{1} \hat{W}_{2,a}^{(h)}(0) = 0$ and as a consequence $|\sum_{h=k}^{1} \hat{W}_{2,a}^{(h)}(0)| \leq C|\lambda|\gamma^{(1+\vartheta)k}$ as $|\hat{W}_{2,a}^{(h)}(0)| \leq C|\lambda|\gamma^{(1+\vartheta)h}$. On the other hand $|\hat{W}_{2,b}^{(h)}(0)| \leq C|\lambda||r|\gamma^{\vartheta h}$ so that

$$\gamma^{h-1}\mu_{h-1} = \gamma^h \mu_h + \hat{W}_2^{(h)}(0) \tag{99}$$

hence $\gamma^{h-1}\mu_{h-1} = \sum_{h=h}^{1} \hat{W}_2^{(k)}(0)$ and $|\mu_h| \leq C|\lambda|$.

It remains to prove that $\sum_{h=-\infty}^{1} \hat{W}_{2,a}^{(h)}(0) = 0$. This can be checked noting that in the $r = 0$ case it is more natural to consider the following ultraviolet regularization, instead of (16) at $\beta = \infty$

$$g(\mathbf{x}) = \vartheta_M(x_0) \int_{-\pi}^{\pi} \mathrm{d}k e^{ikx + \varepsilon(k)x_0} \tag{100}$$

with $\varepsilon(k) = \cos(k) - 1$ and $\vartheta_M(x_0)$ is a smooth function with support in $\gamma^{-M}, +\infty$; note that $g(\mathbf{x})$ verifies (24). We can write $g(\mathbf{x}) = g^{(u.v.)}(\mathbf{x}) + g^{(i.r.)}(\mathbf{x})$ with $g^{(u.v.)}(\mathbf{x}) = h(x_0)g(\mathbf{x})$ and $g^{(i.r.)}(\mathbf{x}) = (1 - h(x_0))g(\mathbf{x})$, and $h(x_0)$ a smooth function $= 1$ is $|x_0| < 1$ and $= 0$ if $|x_0| > \gamma$. The integration of the ultraviolet part can be done as in section 3 of [11], writing $\vartheta_M(x_0)$ as sum of compact support functions. After that, the limit $M \to \infty$ can be taken, and we can write $g^{(i.r.)}(\mathbf{x}) = \sum_{h=-\infty}^{-1} g^{(h)}(\mathbf{x})$ with

$$g^{(h)}(\mathbf{x}) = \vartheta(x_0)(1 - h(x_0)) \int_{-\pi}^{\pi} \mathrm{d}k c_h(k) e^{ikx + \varepsilon(k)x_0} \tag{101}$$

with $c_h(k)$ a smooth function non-vanishing for $\pi\gamma^{h-1} \leq |k| \leq \pi\gamma^{h+1}$; note that $g^{(h)}(\mathbf{x})$ verifies (68), and the integration of the infrared scales is essentially identical to the one described in this sections. Once all scales are integrated out, we obtain kernels $W_{n,m}^{(-\infty)}$ coinciding with the ones obtained before; however, with this choice of the ultraviolet cutoff, $W_{2,0}^{(-\infty)} \equiv 0$ is an immediate consequence of the presence of the $\vartheta_M(x_0)$ in the propagator. Indeed the kernels can be written as sum over Feynman graphs, which contain surely a closed fermionic loop or a tadpole (the interaction is local in time). $\qquad \square$

2.3. The 2-point Schwinger Function in the Insulating Phase

In the case $r = 0$ we have $h^* = -\infty$ and the integration considered in this section conclude the construction of the effective potential. Similarly, if $r < 0$ and $|\lambda|$ is small then $g^{(<h^*)} \equiv 0$, so that again the construction of the effective potential is concluded by the integration on scale h^*.

In both case the analysis described above can be easily extended to take into account the external fields, that is $\phi \neq 0$ (see for instance section 3.4 of [4] for details in a similar case). The 2-point Schwinger function can be written as, if we define $h_{\mathbf{k}} = \min\{h : \hat{g}(\mathbf{k}) \neq 0\}$

$$\hat{S}(\mathbf{k}) = \sum_{j=h_{\mathbf{k}}}^{h_{\mathbf{k}}+1} Q_{\mathbf{k}}^{(j)} \hat{g}^{(j)}(\mathbf{k}) Q_{\mathbf{k}}^{(j)} - \sum_{j=h_{\mathbf{k}}}^{h_{\mathbf{k}}+1} G^{(j)}(\mathbf{k}) \hat{W}_{2,0}^{(j-1)}(\mathbf{k}) G^{(j)}(\mathbf{k}) \qquad (102)$$

where from (81) $|\hat{W}_{2,0}^{(j-1)}(\mathbf{k})| \leq C|\lambda|\gamma^{(1+\vartheta)j}$, $Q_{\mathbf{k}}^{(h)}$ is defined inductively by the relation $Q^{(1)} = 1$ and

$$Q_{\mathbf{k}}^{(h)} = 1 - \hat{W}_{2,0}^{(h)}(\mathbf{k}) g^{(h+1)}(\mathbf{k}) Q_{\mathbf{k}}^{(h+1)} \qquad (103)$$

and

$$G^{(h+1)}(\mathbf{k}) = \sum_{k=h=1}^{1} g^{(k)}(\mathbf{k}) Q_{\mathbf{k}}^{(k)} \qquad (104)$$

so that

$$|Q_{\mathbf{k}}^{(h)} - 1| \leq C|\lambda|\gamma^{\vartheta h} \qquad |\hat{G}^{(j)}(\mathbf{k})| \leq C|\lambda|\gamma^{-\vartheta h} \qquad (105)$$

Using that

$$\hat{g}^{(h)}(\mathbf{k}) = \frac{f_h(\mathbf{k})}{-ik_0(1 + z_{-\infty}) + (1 + \alpha_{-\infty})\frac{k^2}{2}} + \hat{r}^{(h)}(\mathbf{k}) \qquad (106)$$

where

$$|\hat{r}^{(h)}(\mathbf{k})| \leq C\gamma^{(1-\vartheta)h} \qquad (107)$$

so that (14) follows.

3. Renormalization Group Integration: The Second Regime in the Metallic Phase

3.1. The Anomalous Integration

We have now to consider the integration of the scales with $h < h^*$, that is

$$e^{-\mathscr{W}(0)} = e^{-\beta L F_{h^*}} \int P(d\psi^{(\leq h^*)}) e^{-\mathscr{V}^{(h^*)}(\psi^{(\leq h^*)})} \qquad (108)$$

where $P(d\psi^{(\leq h^*)})$ has propagator given by

$$g^{(\leq h^*)}(\mathbf{x}) = \int d\mathbf{k} \frac{\chi_{\leq h^*}(\mathbf{k})}{-ik_0(1 + z_{h^*}) + (1 + \alpha_{h^*})(\cos k - 1) + r + \gamma^{h^*}\mu_{h^*}} \qquad (109)$$

with $z_{h^*}, \alpha_{h^*}, \nu_{h^*} = O(\lambda)$.

The denominator of the propagator (109) vanishes in correspondence of the two Fermi momenta and we need a multiscale decomposition. It is convenient to rewrite (108) in the following way

$$\int P(\mathrm{d}\psi^{(\leq h^*)})\mathrm{e}^{-\mathscr{V}^{(h^*)}(\psi^{(\leq h^*)})}$$

$$= \int \widetilde{P}(\mathrm{d}\psi^{(\leq h^*)})\mathrm{e}^{-\mathscr{V}^{(h^*)}(\psi^{(\leq h^*)})-\gamma^{h^*}\nu_{h^*}\int \mathrm{d}\mathbf{x}\psi_{\mathbf{x}}^{(\leq h^*)+}\psi_{\mathbf{x}}^{(\leq h^*)-}} \quad (110)$$

where $\widetilde{P}(\mathrm{d}\psi^{(\leq h^*)})$ has propagator

$$g^{(\leq h^*)}(\mathbf{x}) = \int \mathrm{d}\mathbf{k}\,\mathrm{e}^{i\mathbf{k}\mathbf{x}}\frac{\chi_{\leq h^*}(\mathbf{k})}{-ik_0(1+z_{h^*})+(1+\alpha_{h^*})(\cos k - \cos p_F)} \quad (111)$$

and

$$(1+\alpha_{h^*})\cos p_F = (1+\alpha_{h^*}) - r - \gamma^{h^*}\mu_{h^*} + \gamma^{h^*}\nu_{h^*}. \quad (112)$$

Observe that, assuming that also $\nu_{h^*} \leq K|\lambda|$, then we have

$$\sqrt{\frac{r}{2}} \leq p_F \leq 2\sqrt{2r} \quad (113)$$

for λ such that $K|\lambda| \leq 1/32$. The strategy of the analysis is the following:

(a) we will perform a multiscale analysis of (108). In this analysis we will have to choose $\nu_{h^*} = O(\lambda)$ as function of p_F and λ to obtain a convergent expansion.

(b) at the end of the above construction we will use (112) to obtain the Fermi momentum p_F as function of λ and r.

We can now write

$$\chi_{\leq h^*}(\mathbf{k}) = \chi_{\leq h^*,1}(\mathbf{k}) + \chi_{\leq h^*,-1}(\mathbf{k})$$

where

$$\chi_{\leq h^*,\omega}(\mathbf{k}) = \widetilde{\vartheta}\left(\omega\frac{k}{p_F}\right)\chi_{\leq h^*}(\mathbf{k})$$

where $\omega = \pm 1$, $\widetilde{\vartheta}$ is a smooth function such that $\widetilde{\vartheta}(k) = 1$ for $k > \frac{1}{2}$ and $\widetilde{\vartheta}(k) = 0$ for $k < -\frac{1}{2}$ and

$$\widetilde{\vartheta}(k) + \widetilde{\vartheta}(-k) = 1$$

for every k. Thus, $\widetilde{\vartheta}\left(\frac{k}{p_F}\right)$ is equal to 1 in a neighbor of p_F and 0 in a neighbor of $-p_F$. Clearly $\chi_{\leq h^*,\pm 1}(\mathbf{k})$ is a smooth, compact support function and it allows us to write

$$g^{(\leq h^*)}(\mathbf{x}) = \sum_{\omega=\pm 1} \mathrm{e}^{i\omega p_F x}g_{\omega}^{(\leq h^*)}(\mathbf{x}) \quad (114)$$

where

$$g_{\omega}^{(\leq h^*)}(\mathbf{x}) = \int \mathrm{d}\mathbf{k}\,\mathrm{e}^{i(\mathbf{k}-\omega\mathbf{p}_F)\mathbf{x}}\frac{\chi_{\leq h^*,\omega}(\mathbf{k})}{-ik_0(1+z_{h^*})+(1+\alpha_{h^*})(\cos k - \cos p_F)} \quad (115)$$

with $\mathbf{p}_F = (0, p_F)$.

We observe that, if the running coupling constants were not present in the cutoff function $\chi_{\leq h}$, we could have used as a quasi-particle cutoff function

$$\widetilde{\chi}_{\leq h^*,\pm 1}(\mathbf{k}) = \vartheta\,(\pm k)\,\chi_{\leq h^*}(\mathbf{k})$$

where $\vartheta(k) = 1$ if $k > 0$ and $\vartheta(k) = 0$ if $k < 0$. Indeed, thanks to (66), this would have made essentially no difference. On the other hand, thanks to (113) and (67), we have that $\chi_{\leq h^*,\pm 1}$ differs from $\widetilde{\chi}_{\leq h^*,\pm 1}$ only for a finite number (not depending on r) of scales so that this does not modify our qualitative picture. Finally notice that the argument of $\widetilde{\vartheta}$ is not scaled with γ^{-h} but only with $p_F^{-1} = O(\gamma^{-\frac{h^*}{2}})$.

The multiscale integration is done exactly as in [4]. The localization operation is defined in the following way

$$\mathscr{L}_2 \int d\underline{\mathbf{x}} W_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \prod_{i=1}^{4} \psi_{\mathbf{x}_i,\omega_i}^{\varepsilon_i} = \hat{W}_4(0) \int d\mathbf{x} \psi_{\mathbf{x},1}^{+} \psi_{\mathbf{x},1}^{-} \psi_{\mathbf{x},-1}^{+} \psi_{\mathbf{x},-1}^{-}$$

$$\mathscr{L}_2 \int d\underline{\mathbf{x}} W_2(\mathbf{x}_1, \mathbf{x}_2) \psi_{\mathbf{x}_1,\omega}^{+} \psi_{\mathbf{x}_2,\omega}^{-} = \hat{W}_2(0) \int \psi_{\mathbf{x},\omega}^{+} \psi_{\mathbf{x},\omega}^{-} d\mathbf{x}$$

$$+ \partial_1 \hat{W}_2(0) \int \bar{\psi}_{\mathbf{x},\omega}^{+} \Delta_1 \psi_{\mathbf{x},\omega}^{-} d\mathbf{x} + \partial_0 \hat{W}_2(0) \int \psi_{\mathbf{x},\omega}^{+} \partial_0 \psi_{\mathbf{x},\omega}^{-} d\mathbf{x} \qquad (116)$$

where

$$\bar{\Delta}_1 f(\mathbf{x}) = 2 \int d\mathbf{k}(\cos k - \cos p_F) e^{i\mathbf{k}\mathbf{x}} \hat{f}(\mathbf{k}) \quad \text{if} \quad f(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \hat{f}(\mathbf{k}).$$

Note that in the kernels W_l are included the oscillating factors $e^{i\omega p_F x}$ coming form (114).

After the integration of the scale $\psi^{(h^*)}, \ldots \psi^{(h)}$ we get

$$e^{-\mathscr{W}(0)} = e^{-\beta L F_h} \int P_{Z_h}(d\psi^{(\leq h)}) e^{-\mathscr{V}^{(h)} l\left(\sqrt{Z_h}\psi^{(\leq h)}\right)} \qquad (117)$$

where $P_{Z_h}(d\psi^{(\leq h)})$ is the Grassmann integration with propagator $\frac{g_\omega^{(\leq h)}}{Z_h}$ where

$$g_\omega^{(\leq h)}(\mathbf{x}) = \int d\mathbf{k} e^{i(\mathbf{k}-\omega \mathbf{p}_F)} \frac{\chi_{\leq h,\omega}(\mathbf{k})}{-(1+z_{h^*})ik_0 + (1+\alpha_{h^*})(\cos k - \cos p_F)} \qquad (118)$$

where

$$\chi_{\leq h,\omega}(\mathbf{k}) = \widetilde{\vartheta}\left(\omega \frac{k}{p_F}\right) \chi_0(a_0 \gamma^{-h} |(1+z_{h^*})ik_0 - (1+\alpha_{h^*})(\cos k - \cos p_F)|). \qquad (119)$$

Observe that in the denominator of (118) and in (119) we have the running constant α_{h^*} and z_{h^*}.

We can now write

$$\int P_{Z_h}(d\psi^{(\leq h)}) e^{-\mathscr{V}^{(h)}\left(\sqrt{Z_h}\psi^{(\leq h)}\right)} = \int \widetilde{P}_{Z_{h-1}}(d\psi^{(\leq h)}) e^{-\widetilde{\mathscr{V}}^{(h)}\left(\sqrt{Z_h}\psi^{(\leq h)}\right)}$$

where

$$\mathscr{L}_2 \widetilde{\mathscr{V}}^{(h)}(\psi) = l_h \int d\mathbf{x} \psi^+_{\mathbf{x},1} \psi^-_{1,\mathbf{x}} \psi^+_{\mathbf{x},-1} \psi^-_{-1,\mathbf{x}} + (a_h - z_h) \sum_\omega \int d\mathbf{x} \psi^+_{\omega,\mathbf{x}} \Delta_1 \psi^-_{\omega,\mathbf{x}}$$

$$+ n_h \int d\mathbf{x} \psi^+_{\omega,\mathbf{x}} \psi^-_{\omega,\mathbf{x}} \tag{120}$$

while $\widetilde{P}_h(d\psi^{(\le h)})$ is the integration with propagator identical to (118) but with $\chi_{\le h,\omega}(\mathbf{k})$ replaced by $\frac{\chi_{\le h,\omega}(\mathbf{k})}{\widetilde{Z}_{h-1}(\mathbf{k})}$ with

$$\widetilde{Z}_{h-1}(\mathbf{k}) = Z_h + \chi_{\le h,\omega}(\mathbf{k}) Z_h z_h.$$

Setting $Z_{h-1} = \widetilde{Z}_{h-1}(0)$, we can finally write

$$\int P_{Z_h}(d\psi^{(\le h)}) e^{-\mathscr{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})}$$

$$= e^{-\beta L e_h} \int P_{Z_{h-1}}(d\psi^{(\le h-1)}) \int \widetilde{P}_{Z_{h-1}}(d\psi^{(h)}) e^{-\bar{\mathscr{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})} \tag{121}$$

where $\widetilde{P}_{Z_{h-1}}$ is the integration with propagator $\frac{\widetilde{g}^{(h)}_\omega}{Z_{h-1}}$

$$\widetilde{g}^{(h)}_\omega(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{x}(\mathbf{k}-\omega\mathbf{p}_F)} \frac{\widetilde{f}_{h,\omega}(\mathbf{k})}{-(1+z_{h*})ik_0 + (1+\alpha_{h*})(\cos k - \cos p_F)}$$

where

$$\widetilde{f}_{h,\omega}(\mathbf{k}) = Z_{h-1} \left[\frac{\chi_{\le h,\omega}(\mathbf{k})}{\widetilde{Z}_{h-1}(\mathbf{k})} - \frac{\chi_{\le h-1,\omega}(\mathbf{k})}{Z_{h-1}} \right].$$

Finally we have

$$\bar{\mathscr{V}}^{(h)}(\psi^{(\le h)}) = \widetilde{\mathscr{V}}^{(h)} \left(\sqrt{\frac{Z_h}{Z_{h-1}}} \psi^{(\le h)} \right) \tag{122}$$

so that

$$\mathscr{L}_2 \widetilde{\mathscr{V}}^h = \lambda_h \int d\mathbf{x} \psi^+_{\mathbf{x},1} \psi^-_{1,\mathbf{x}} \psi^+_{\mathbf{x},-1} \psi^-_{-1,\mathbf{x}} + \delta_h \sum_\omega \int d\mathbf{x} \psi^+_{\omega,\mathbf{x}} \Delta_1 \psi_{\omega,\mathbf{x}}$$

$$+ \gamma^h \nu_h \sum_\omega \int d\mathbf{x} \psi^+_{\omega,\mathbf{x}} \psi_{\omega,\mathbf{x}} \tag{123}$$

with

$$\gamma^h \nu_h = \frac{Z_h}{Z_{h-1}} n_h \qquad \delta_h = \frac{Z_h}{Z_{h-1}} (a_h - z_h) \qquad \lambda_h = \left(\frac{Z_h}{Z_{h-1}} \right)^2 l_h$$

We can now integrate the field $\psi^{(h)}$

$$\int \widetilde{P}_{Z_{h-1}}(d\psi^{(h)}) e^{-\bar{\mathscr{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})} = e^{-\beta L \widetilde{e}_h - \mathscr{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)})} \tag{124}$$

so that the procedure can be iterated.

We can now state the following Lemma whose proof is reported in Appendix B.

Lemma 3.1. *For $h \le h^*$, every N and λ small enough we have*

$$|\partial_0^{n_0} \partial_1^{n_1} \widetilde{g}_\omega^{(h)}(\mathbf{x})| \le C_N \frac{v_F^{-1} \gamma^h}{1 + [\gamma^h |x_0| + v_F^{-1} \gamma^h |x|]^N} \gamma^{h(n_0 + n_1)} v_F^{-n_1} \quad (125)$$

with $v_F = \sin(p_F) = O(r^{\frac{1}{2}})$.

Again the effective potential can be written as a sum over trees similar to the previous ones but with the following modifications:

1. We associate a label $h \le h^*$ with the root.
2. With each endpoint v we associate one of the monomials contributing to $\mathscr{R}_2 \mathscr{V}^{(h^*)}(\psi^{(\le h_v - 1)})$ or one of the terms contributing to $\mathscr{L}_2 \mathscr{V}^{(h_v)}(\psi^{(\le h_v - 1)})$.

The main result of this section is the following Lemma.

Lemma 3.2. *Assume that*

$$|\lambda_k|, |\delta_k| \le C v_F |\lambda| \quad |\nu_k| \le C|\lambda| \quad (126)$$

than there exists a constants $\lambda_0 > 0$, independent of β, L and r, such that, for $h < h^*$, the kernels $W_l^{(h)}$ are analytic functions of λ for $|\lambda| \le \lambda_0$. Moreover, they satisfy

$$\frac{1}{\beta L} \int d\mathbf{x}_1 \cdots d\mathbf{x}_l |W_l^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_l)| \le \gamma^{h\left(2 - \frac{l}{2}\right)} v_F^{\frac{l}{2} - 1} (C|\lambda|)^{\max(1, l-1)}. \quad (127)$$

Proof. The proof of this Lemma follows closely the line of [4]. The only major difference is the presence of the small factors in (125). We will report only the modification of the proof needed to deal with those factors.

We start noting that the analogous of the bound (84) becomes

$$\frac{1}{\beta L} \int d\mathbf{x}_1 \cdots d\mathbf{x}_l |W_l^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_l)|$$

$$\le \sum_{n \ge 1} \sum_{\tau \in \mathscr{T}_{h,n}} \sum_{\substack{\mathbf{P} \in \mathscr{P}_\tau \\ |P_{v_0}| = l}} \sum_{T \in \mathbf{T}} C^n \left[\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{h_v \left(\sum_{i=1}^{s_v} \frac{|P_{v_i}|}{2} - \frac{|P_v|}{2} - 2(s_v - 1) \right)} \right]$$

$$\left[\prod_{v \text{ not e.p.}} \left(\frac{1}{v_F} \right)^{\sum_{i=1}^{s_v} \frac{|P_{v_i}|}{2} - \frac{|P_v|}{2} - (s_v - 1)} \right] \left[\prod_{v \text{ not e.p.}} \gamma^{-(h_v - h_{v'}) z_2(P_v)} \right]$$

$$\left[\prod_{v \text{ e.p.}; v \in I^R, |I_v| \ge 6} |\lambda| \gamma^{h^* \left(\frac{3}{2} - \frac{|I_v|}{4} \right)} \right]$$

$$\left[\prod_{v \text{ e.p.}; v \in I^R, |I_v| = 2,4} |\lambda| \gamma^{h^* \left(\frac{3}{2} - \frac{|I_v|}{4} \right) + z_2(P_v)(h_{v'} - h^*)} \right]$$

$$\left[\prod_{v \text{ e.p.}; v \in I^\lambda} |\lambda| v_F \right] \left[\prod_{v \text{ e.p.}; v \in I^{\nu, \delta}} |\lambda| \gamma^{h_{v'}} \right] \left[\prod_{i=1}^n C^{p_i} \right] \quad (128)$$

where:

1. the last factor keeps into account the presence of the factors Z_h/Z_{h-1};

2. the factor $\left(\frac{1}{v_F}\right)^{\frac{|P_{v_i}|}{2} - \frac{|P_v|}{2} - (s_v - 1)}$ comes from the bound on the Gram determinant and the fact that $|g^h(\mathbf{x})| \leq \frac{\gamma^h}{v_F}$;

3. $z_2(P_v) = 1$ for $|P_v| = 4$ and $z_2(p_v) = 2$ for $|P_v| = 2$;

4. I^R is the set of endpoints associated to $\mathscr{RV}^{(h^*)}$ and the factor $\gamma^{h^*\left(\frac{3}{2} - \frac{|I_v|}{4}\right)}$ comes from the bound (81);

5. I^λ is the set of endpoints associated to λ_k and the factor v_F comes from (126);

6. I^δ is the set of endpoints associated to δ_k and the derivative in (123) produces an extra $\gamma^{h_{v'}}/v_F$;

7. I^ν is the set of endpoints associated to ν_k and the factor $\gamma^{h_{v'}}$ comes from (123).

Proceeding like in the proof of Lemma 2.2 using (87) we get

$$
\frac{1}{\beta L} \int d\mathbf{x}_1 \cdots d\mathbf{x}_l |W_l^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_l)|
$$

$$
\leq \sum_{n \geq 1} \sum_{\tau \in \mathscr{T}_{h,n}} \sum_{\substack{\mathbf{P} \in \mathscr{P}_\tau \\ |P_{v_0}| = l}} \sum_{T \in \mathbf{T}} C^n \gamma^{h\left(2 - \frac{1}{2}|P_{v_0}| + \frac{1}{2}|I_{v_0}| - 2n\right)}
$$

$$
\left[\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{(h_v - h_{v'})\left(2 - \frac{|P_v|}{2} + \frac{|I_v|}{2} - 2n(v) + z_2(P_v)\right)} \right]
$$

$$
\left[\prod_{v \text{ e.p.}; v \in I^R} |\lambda| \gamma^{h_{v'}\left(\frac{3}{2} - \frac{|I_v|}{4}\right)} \right] \left[\prod_{v \text{ e.p.}; v \in I^\lambda} |\lambda| v_F \right]
$$

$$
\left[\prod_{v \text{ e.p.}; v \in I^{\nu,\delta}} |\lambda| \gamma^{h_{v'}} \right] \left[\prod_{i=1}^{n} C^{p_i} \right]
$$

$$
\left[\prod_{v \text{ not e.p.}} \left(\frac{1}{v_F}\right)^{\left(\sum_{i=1}^{s_v} \frac{|P_{v_i}|}{2} - \frac{|P_v|}{2} - (s_v - 1)\right)} \right]. \tag{129}
$$

Finally using (89) we arrive to

$$
\frac{1}{\beta L} \int d\mathbf{x}_1 \cdots d\mathbf{x}_l |W_l^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_l)|
$$

$$
\leq \sum_{n \geq 1} \sum_{\tau \in \mathscr{T}_{h,n}} \sum_{\substack{\mathbf{P} \in \mathscr{P}_\tau \\ |P_{v_0}| = l}} \sum_{T \in \mathbf{T}} C^n \gamma^{h\left(2 - \frac{1}{2}|P_{v_0}|\right)}
$$

$$
\left[\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{(h_v - h_{v'})\left(2 - \frac{|P_v|}{2} + z_2(P_v)\right)} \right]
$$

$$\left[\prod_{v \text{ e.p.}, v\in I^R} \gamma^{h_{v'}\left(-\frac{1}{2}+\frac{|I_v|}{4}\right)}\right] \left[\prod_{v \text{ e.p.}; v\in I^\lambda} |\lambda| v_F\right] \left[\prod_{v \text{ e.p.}; v\in I^{\nu,\delta}} |\lambda|\right] \left[\prod_{i=1}^n C^{p_i}\right]$$

$$\left[\prod_{v \text{ not e.p.}} \left(\frac{1}{v_F}\right)^{\left(\sum_{i=1}^{s_v} \frac{|P_{v_i}|}{2}-\frac{|P_v|}{2}-(s_v-1)\right)}\right]. \tag{130}$$

Because $\gamma^{h_v'} \le \gamma^{h^*} \le v_F^2$ and $|I_v| \ge 2$ we have

$$\gamma^{h_{v'}\left(-\frac{1}{2}+\frac{|I_v|}{4}\right)} \le v_F^{-1+\frac{|I_v|}{2}}$$

so that

$$\frac{1}{\beta L} \int d\mathbf{x}_1 \cdots d\mathbf{x}_l |W_l^{(h)}(\mathbf{x}_1,\ldots,\mathbf{x}_l)|$$

$$\le \sum_{n\ge 1} \sum_{\tau\in\mathscr{T}_{h,n}} \sum_{\substack{\mathbf{P}\in\mathscr{P}_\tau \\ |P_{v_0}|=l}} \sum_{T\in\mathbf{T}} C^n \gamma^{h\left(2-\frac{1}{2}|P_{v_0}|\right)}$$

$$\left[\prod_{v \text{ not e.p.}} \frac{1}{s_v!}\gamma^{(h_v-h_{v'})\left(2-\frac{|P_v|}{2}+z_2(P_v)\right)}\right]$$

$$\left[\prod_{v \text{ e.p.}, v\in I^R, I^\lambda} v_F^{-1+\frac{|I_v|}{2}}\right] \left[\prod_{v \text{ not e.p.}} \left(\frac{1}{v_F}\right)^{\left(\sum_{i=1}^{s_v}\frac{|P_{v_i}|}{2}-\frac{|P_v|}{2}-(s_v-1)\right)}\right]$$

$$|\lambda|^n \left[\prod_{i=1}^n C^{p_i}\right]. \tag{131}$$

For $v\in I^\delta, I^\nu$, one has $|I_v|=2$ so that $v_F^{-1+\frac{|I_v|}{2}}=1$, and we can write

$$\left[\prod_{v \text{ e.p.}, v\in I^R, I^\lambda} v_F^{-1+\frac{|I_v|}{2}}\right] = \left[\prod_{v \text{ e.p.}} v_F^{-1+\frac{|I_v|}{2}}\right] = v_F^{-n+\sum_{v \text{ e.p}}\frac{|I_v|}{2}}. \tag{132}$$

Using that

$$\sum_v (s_v-1) = n-1 \qquad\qquad \sum_{v \text{ e.p.}} |I_v| = l + \sum_v \sum_{i=1}^{s_v} (|P_{v_i}|-|P_v|)$$

we get

$$\prod_{v \text{ e.p.}} v_F^{-1} \prod_{v \text{ not e.p.}} \left(\frac{1}{v_F}\right)^{-(s_v-1)} = v_F^{-1}$$

$$\prod_{v \text{ e.p.}} v_F^{\frac{|I_v|}{2}} \prod_{v \text{ not e.p.}} \left(\frac{1}{v_F}\right)^{\left(\sum_{i=1}^{s_v}\frac{|P_{v_i}|}{2}-\frac{|P_v|}{2}\right)} = v_F^{\frac{l}{2}}. \tag{133}$$

Collecting these estimates we get

$$\frac{1}{\beta L} \int \mathrm{d}\mathbf{x}_1 \cdots \mathrm{d}\mathbf{x}_l |W_l^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_l)|$$

$$\leq v_F^{\frac{l}{2}-1} \sum_{n \geq 1} \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\substack{\mathbf{P} \in \mathcal{P}_\tau \\ |P_{v_0}|=l}} \sum_{T \in \mathbf{T}} C^n \gamma^{h\left(2 - \frac{1}{2}|P_{v_0}|\right)}$$

$$\left[\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{(h_v - h_{v'})\left(2 - \frac{|P_v|}{2} + z_2(P_v)\right)} \right] |\lambda|^n \left[\prod_{i=1}^n C^{p_i} \right]. \tag{134}$$

Performing the sums as in the previous section we prove (127).   □

Remark.   • Observe that, for $h \geq h^*$, bound (81) says that the L_1 norm of the effective potential is $O(\gamma^{h(3/2-l/4)})$ while, for $h \leq h^*$, bound (127) says that the L_1 norm of the effective potential is $O(\gamma^{h(2-l/2)} v_F^{\frac{l}{2}-1})$; the two bounds coincide of course at $h \simeq h^*$ since $\gamma^{h^*} \sim r$, $v_F \sim \sqrt{r}$ so that $r^{(2-l/2)} r^{\frac{l}{4}-\frac{1}{2}} = r^{\frac{3}{2}-\frac{l}{4}}$.

• The fact that the Fermi velocity vanishes as r approaches 0 produces the "dangerous" factor $\left(\frac{1}{v_F}\right)^{\frac{|P_{v_i}|}{2} - \frac{|P_v|}{2} - (s_v-1)}$ in (128) which is diverging as $r \to 0$. This is compensated by the extra factors of v_F associated to the difference between the scaling dimensions the first and second regime, that is

$$\left[\prod_{v \text{ e.p.}} \gamma^{h_{v'}\left(\frac{3}{2} - \frac{|I_v|}{4}\right)} \right] = \left[\prod_{v \text{ e.p.}} \gamma^{h_{v'}\left(2 - \frac{|I_v|}{2}\right)} \right] \left[\prod_{v \text{ e.p.}} \gamma^{h_{v'}\left(-\frac{1}{2} + \frac{|I_v|}{4}\right)} \right]. \tag{135}$$

3.2. The Flow of the Running Coupling Constants

We now prove by induction that, for $h \leq h^*$ and $\vartheta = \frac{1}{4}$ we have

$$|\lambda_h| \leq C|\lambda| r^{\frac{1}{2}+\vartheta}, \quad |\delta_h| \leq C|\lambda| r^{\frac{1}{2}+\vartheta} \quad |\nu_h| \leq C|\lambda| \gamma^{\vartheta h}. \tag{136}$$

First we check that (126) is true for $h = h^*$. By definition of the $\mathscr{L}_2$ operation

$$\lambda_{h^*} = \lambda \left[\hat{v}(0) - \hat{v}(2p_F)\right] + O\left(\lambda^2 \gamma^{h^*\left(\frac{1}{2}+\vartheta\right)}\right) \tag{137}$$

where the second term in the r.h.s comes from (81); as $\hat{v}(k)$ is even the first term is $O(r)$ so that surely λ_{h^*} vanishes as $O\left(r^{\frac{1}{2}+\vartheta}\right)$. Moreover, from (81), taking into account that a derivative ∂_1 gives an extra $\gamma^{-h/2}$, that is

$$\int \mathrm{d}\mathbf{x} |\partial_1 W_2^{(h^*)}(\mathbf{x})| \leq C|\lambda| \gamma^{h^*\left(\frac{1}{2}+\vartheta\right)} \tag{138}$$

we get

$$|\delta_{h^*}| \leq C\gamma^{h\left(\frac{1}{2}+\vartheta\right)} |\lambda| \leq C|\lambda| r^{\frac{1}{2}+\vartheta}. \tag{139}$$

The flow of ν_h is given by

$$\nu_{h-1} = \gamma \nu_h + \beta_\nu^{(h)}(\vec{v}_h, \ldots, \vec{v}_0) \tag{140}$$

where $\vec{v}_h = (\lambda_h, \delta_h, \nu_h)$. We can decompose the propagator as

$$\widetilde{g}_\omega^{(h)}(\mathbf{x}) = g_{\omega,L}^{(h)}(\mathbf{x}) + r_\omega^{(h)}(\mathbf{x}) \tag{141}$$

where

$$g_{\omega,L}^{(h)}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{\widetilde{f}_h(\mathbf{k})}{-ik_0 + \omega v_F k} \tag{142}$$

and $\widetilde{f}_h$ has support contained in $C\gamma^{h-1} \le \sqrt{k_0^2 + v_F^2 k^2} \le C\gamma^{h+1}$. Moreover, for every N, we have

$$|r_\omega^{(h)}(\mathbf{x})| \le \left(\frac{\gamma^h}{v_F}\right)^3 \frac{C_N}{1 + \gamma^h(|x_0| + v_F^{-1}|x|)^N} \tag{143}$$

that is the bound for $r_\omega^{(h)}(\mathbf{x})$ has an extra factor $\gamma^{2h}/v_F^2 \le \gamma^h$ with respect to the bound (125) for $\widetilde{g}_\omega^{(h)}(\mathbf{x})$.

In the expansion for $\beta_\nu^{(h)}$ studied in the previous subsection, we can decompose every propagator as in (141) and collect all the term that contains only $g_{L,\omega}^{(h)}$ and that come from trees with no endpoints associated to $\mathscr{R}\mathscr{V}^{(h^*)}$; this sum vanish due to parity. Therefore, $\beta_\nu^{(h)} = O(\lambda\gamma^{\vartheta h})$ and by iteration

$$\nu_{h-1} = \gamma^{-h+h^*}\left[\nu_{h^*} + \sum_{k=h}^{h^*} \gamma^{k-h^*}\beta_\nu^{(k)}\right]. \tag{144}$$

Thus, we can choose ν_{h^*} so that

$$\nu_{h^*} = -\sum_{k=-\infty}^{h^*} \gamma^{k-h^*}\beta_\nu^{(k)}. \tag{145}$$

This implies that

$$\nu_{h-1} = \gamma^{-h+h^*}\left[-\sum_{k=-\infty}^{h} \gamma^{k-h^*}\beta_\nu^{(k)}\right] \tag{146}$$

and $|\nu_h| \le C|\lambda|\gamma^{\vartheta h}$.

We now study the flow equations for λ_h and δ_h with $h < h^*$

$$\lambda_{h-1} = \lambda_h + \beta_\lambda^{(h)}(\vec{v}_h, \ldots, \vec{v}_0)$$
$$\delta_{h-1} = \delta_h + \beta_\delta^{(h)}(\vec{v}_h, \ldots, \vec{v}_0) \tag{147}$$

where we have redefined δ_0 as to include the sum $\widetilde{\delta}_0$ of the terms $O(\lambda)$, which satisfies

$$|\widetilde{\delta}_0| \le C\left|\int d\mathbf{k} k\partial^2 v(\mathbf{k} + (\omega - \omega')\mathbf{p}_F)g_{\overline{\omega}}^{\le h^*}(\mathbf{k})\right| \tag{148}$$

where one derivative over v comes from the $\mathscr{R}_1$ operation and the other from the definition of δ. Observe that

$$|\widetilde{\delta}_0| \le C\sum_{k \le h^*} v_F^{-2}\gamma^{2h} \le C|\lambda|r \tag{149}$$

since $v_F k \le C\gamma^h$ in the support of f_h.

Again we can use (141) and decompose the beta function for $\alpha = \lambda, \delta$ as

$$\beta_\alpha^{(h)}(\vec{v}_h, \ldots, \vec{v}_0) = \bar{\beta}_\alpha^{(h)}(\lambda_h, \delta_h, \ldots, \lambda_0, \delta_0) + \beta_{\alpha,R}^{(h)}(\vec{v}_h, \ldots, \vec{v}_0) \qquad (150)$$

where $\bar{\beta}_\alpha^{(h)}$ contains only propagators $g_{\omega,L}^{(h)}(\mathbf{x})$ and endpoints to which is associated λ_k, δ_k. Therefore, $\beta_{\alpha,R}^{(h)}$ contains either a propagator $r_{\omega,L}^{(h)}(\mathbf{x})$, a ν_k or an irrelevant term. Observe that

1. Terms containing a propagator r_h or a factor ν_h have an extra $\gamma^{\vartheta h}$ in their bounds, therefore, by an argument similar to the one used in (70) (*short memory property*) they can be bounded as $O(v_F \gamma^{\vartheta h})$. The factor v_F comes from the factor $v_F^{l/2-1}$ in (127) when $\alpha = \lambda$, and from the derivative ∂_1 in the case $\alpha = \delta$.
2. The terms containing an irrelevant endpoints associated to a term $\mathscr{R}V^{(h^*)}$ have an extra $\gamma^{\vartheta h^*}$ [coming from (59)] and an extra $\gamma^{\vartheta(h-h^*)}$ for the short memory property; therefore, they can be bound as $O(v_F \gamma^{\vartheta h})$. The origin of the factor v_F is the same as in the previous point.

In conclusion,

$$|\beta_{\alpha,R}^{(h)}| \leq C v_F \lambda^2 \gamma^{\vartheta h}. \qquad (151)$$

From (142) it is easy to see that

$$\begin{aligned}
\bar{\beta}_\lambda^{(h)}(\lambda_h, \delta_h, \ldots, \lambda_0, \delta_0) &= v_F \hat{\beta}_\lambda^{(h)}\left(\frac{\lambda_h}{v_F}, \frac{\delta_h}{v_F}, \ldots, \frac{\lambda_0}{v_F}, \frac{\delta_0}{v_F}\right) \\
\bar{\beta}_\delta^{(h)}(\lambda_h, \delta_h, \ldots, \lambda_0, \delta_0) &= v_F \hat{\beta}_\delta^{(h)}\left(\frac{\lambda_h}{v_F}, \frac{\delta_h}{v_F}, \ldots, \frac{\lambda_0}{v_F}, \frac{\delta_0}{v_F}\right)
\end{aligned} \qquad (152)$$

where $\hat{\beta}_\lambda^{(h)}(\lambda_d, \delta_h, \ldots, \lambda, \delta_0)$ is the beta function of a Luttinger model with $v_F = 1$. The following crucial result, called *asymptotic vanishing of the beta function*, has been proved in [5] that

$$|\hat{\beta}_\lambda^{(h)}(\lambda_d, \delta_h, \ldots, \lambda_0, \delta_0)| \leq C \left[\max(|\lambda_k|, |\delta_k|)\right]^2 \gamma^{\vartheta(h-h^*)}. \qquad (153)$$

Assuming by induction that $|\lambda_k|, |\delta_k| \leq 2|\lambda| r^{\frac{1}{2}+\vartheta}$ for $k \geq h$ we get

$$|\bar{\beta}_\alpha^{(h)}(\lambda_h, \delta_h, \ldots, \lambda_0, \delta_0)| \leq 4C v_F \lambda^2 r^{1+2\vartheta} \gamma^{\vartheta h} v_F^{-2} r^{-\vartheta} \leq 4C v_F \lambda^2 \gamma^{\vartheta h} r^\vartheta. \quad (154)$$

Thus,

$$|\lambda_{h-1}| \leq |\lambda_{h^*}| + \sum_{k=h}^{h^*} 4C v_F \lambda^2 \gamma^{\vartheta h} r^\vartheta \leq 2|\lambda| r^{\frac{1}{2}+2\vartheta} \qquad (155)$$

and the same is true for δ_h.

Moreover, we have

$$\frac{Z_{h-1}}{Z_h} = 1 + \beta_z^{(h)} \qquad (156)$$

so that

$$\gamma^\eta = 1 + \beta^{-\infty}\left(\frac{\lambda_{-\infty}}{v_F}\right) \qquad (157)$$

where $\beta^{-\infty}$ is the beta function with $v_F = 1$; therefore,

$$Z_h = \gamma^{-\eta(h-h^*)}(1 + A(\lambda)) \tag{158}$$

with $|A(\lambda)| \leq C|\lambda|$. Observe that $\eta = O(\lambda^2 r^{4\vartheta})$, hence is vanishing as $r \to 0$ as $O(\lambda^2 r)$.

Finally the inversion problem for p_F can be studied as in section 2.9 of [8]. The analysis for the Schwinger function is done in a way similar to the one in section 3 above.

Remark. Note that, from (157), Z_h increases at each iteration step; in the multiscale expansion described in §3.1 one singles out this factor Z_h from the propagator at each integration step and it turns out that such factors are exactly compensated by the couplings of the quartic terms in the effective potential, which increase as $O(\lambda Z_h^2)$, see (122) and (153). This remarkable compensation is established by (117), which was proven in [5,6] by a combination of Ward Identities and Schwinger–Dyson equations.

Appendix A. Grassmann Integrals and Schwinger Functions

It is easy to verify that $S_{L,\beta}(\mathbf{x})$ (5) and $\lim_{M\to\infty} S_{L,\beta}^M(\mathbf{x})$ (26) are order by order equal for $\mathbf{x} \neq (n\beta, 0)$. Indeed they can be written as a power series in terms of λ or $\bar\nu, \lambda$, respectively, and each term of the series can be expressed as a sum of integrals over propagators ($S_{0,L,\beta}(\mathbf{x})$ (7) or $g_{L,\beta}(\mathbf{x})$ (16), respectively) which can be represented by Feynman graphs. The subset of graphs contributing to $S_{L,\beta}(\mathbf{x})$ (5) and with no tadpoles coincides with the graphs contributing to $\lim_{M\to\infty} S_{\beta,L}^M(\mathbf{x})$ (26) and no vertices $\bar\nu$. The integrands are different, as the propagators $S_{0,L\,\beta}(\mathbf{x})$ (7) and $g_{L,\beta}(\mathbf{x})$ (16) are different for $\mathbf{x} = (n\beta, 0)$. Since this is a zero measure set, the integrals are well defined and coincide. Let us consider the remaining graphs. In the graphs with a tadpole in the expansion for $\lim_{M\to\infty} S_{L,\beta}^M(\mathbf{x})$ (26) there is a factor of the form

$$g_{L,\beta}(\mathbf{x}_1 - \mathbf{x})\nu_T g_{L,\beta}(\mathbf{x} - \mathbf{x}_2) , \quad \nu_T = -\lambda\hat{v}(0)\left[\frac{S_{0,L,\beta}(0^+, 0) + S_{0,L,\beta}(0^-, 0)}{2}\right] . \tag{159}$$

On the other hand, given a graph G of this type, there is another graph $\widetilde{G}$, which differs from it only because, in place of the term $\mathscr{V}(\psi)$ which produced the tadpole, there is a vertex $\bar\nu$. If we sum the values of G and $\widetilde{G}$, we get a number which is equal to the value of G, with $-\lambda\hat{v}(0)S_{0,L,\beta}(0^+, 0)$ replacing ν_T, so that the terms coincide with the analogous term in the expansion for $S_{L,\beta}(\mathbf{x})$ (5). Therefore, the perturbative expansion for $S_{L,\beta}(\mathbf{x})$ (5) and $\lim_{M\to\infty} S_{L,\beta}^M(\mathbf{x})$ (26) coincide.

Lemma A.1. *Assume that, for any finite β and L, $S_{\beta,L}^M(\mathbf{x})$ (26) is analytic in $\lambda \in D$, where $D = \{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon_0\}$ with ε independent of β and L, and that it is uniformly convergent as $M \to \infty$. If $\lambda \in D$ and $\mathbf{x} \neq (n\beta, 0)$ then*

$$S_{L\,\beta}(\mathbf{x}) = \lim_{M\to\infty} S_{L,\beta}^M(\mathbf{x}) \tag{160}$$

where $S_{\beta,L}(\mathbf{x})$ is defined in (5) while $S_{L,\beta}^M(\mathbf{x})$ is defined in (26).

Proof. The main point, strictly related with the fact that we are treating a fermionic problem, is that, for L and β finite, $S_{\beta,L}$ (5) is the ratio of the traces of two matrices whose coefficients are entire functions of λ, hence it is the ratio of two entire functions of λ. It follows, in particular, that $S_{\beta,L}$ is the ratio of two functions analytic in D; hence, it may have a singularity in a point $\lambda_0 \in D$ only if $\mathrm{Tr}[e^{-\beta \mathrm{H}}]$ vanishes there, which certainly does not happen in a neighborhood of $\lambda = 0$ small enough (how small possibly depending on L, β). On the other hand, the hypotheses on $S_{\beta,L}^M(\mathbf{x})$ and Weierstrass theorem imply that their limit $M \to \infty$ is analytic in D. Moreover, also the r.h.s. of (160) is analytic in a small neighborhood of $\lambda = 0$ and, as we have explained above, its power expansion in λ coincide with that of $S_{\beta,L}$; hence, the two functions coincide in a disk $\widetilde{D}_{L,\beta}$ with center in $\lambda = 0$ and radius $\varepsilon_{\beta,L}$ possibly vanishing as $\beta, L \to \infty$. However, $S_{\beta,L}$, being the ratio of two functions analytic in D, may have only isolated poles in $D \backslash \widetilde{D}_{L,\beta}$; hence, if E is the set of poles, $S_{\beta,L}$ is analytic in $D \backslash E$ and necessarily coincide with the r.h.s. of (160) in this set, since the two functions coincide in $\widetilde{D}_{L,\beta} \subset D \backslash E$. It follows that, if E were not empty, $S_{\beta,L}$ would be unbounded in $D \backslash E$, while this is not of course true for the other function. $\qquad\square$

Appendix B. Proof of Lemmas 2.1 and 3.1

Proof of Lemma 2.1. We start observing that $f_h(\mathbf{k}) \le \chi_{\le h}(\mathbf{k})$. Moreover, from (57) and the fact that the support of χ_0 is contained in $[0, \gamma]$ we have that, in the support of $f_h(\mathbf{k})$,

$$(1 + z_h)|k_0| \le \gamma^{h+1} a_0 \qquad |(1 + \alpha_h)(\cos k - 1) + r + \gamma^h \mu_h| \le \gamma^{h+1} a_0. \quad (161)$$

Recalling that $a_0 \gamma = 1/2 - r$ and $|r| \le 1/4$ we get $1/4 \le a_0 \gamma \le 3/4$ and assuming $|\lambda|$ so small that $K|\lambda| \le 1/2$ we get

$$|\sin(k)| \le 2\sqrt{1 - \cos^2(k)} \le 2\sqrt{\frac{|r| + K\gamma^h|\lambda| + a_0\gamma^{h+1}}{1 - K|\lambda|}} \le 2\sqrt{3}\gamma^{\frac{h}{2}} \quad (162)$$

so that from (29) we have $|k| \le \sqrt{3}\gamma^{\frac{h}{2}}$. Since the first of the (161) implies $|k_0| \le \frac{3}{2}\gamma^h$, it follows that

$$\int f_h(\mathbf{k}) \mathrm{d}\mathbf{k} \le 6\sqrt{3}\gamma^{\frac{3}{2}h}. \quad (163)$$

Because we clearly have

$$[\gamma^h|x_0| + \gamma^{\frac{h}{2}}|x|]^N |\widetilde{g}^{(h)}(\mathbf{x})| = \sum_{N_1 + N_0 = N} \binom{N}{N_1} |x_0|^{N_0} |x|^{N_1} |\widetilde{g}^{(h)}(\mathbf{x})| \gamma^{h\left(N_0 + \frac{N_1}{2}\right)}$$

to prove the statement for $n_0 = n_1 = 0$ we just need to show that

$$|x_0^{N_0} x^{N_1} \widetilde{g}^{(h)}(\mathbf{x})| \le E\gamma^{h\left(\frac{1}{2} - N_0 - \frac{N_1}{2}\right)} \quad (164)$$

for some suitable constant E. We will use that, for $0 \le x_0 \le \beta/2$, we have

$$x_0 \le \frac{\beta}{2} \sin\left(\frac{\pi}{\beta} x_0\right)$$

and

$$e^{i\frac{\pi}{\beta}x_0}\left(\frac{\beta}{2\pi}\right)^{N_0}\sin\left(\frac{\pi}{\beta}x_0\right)^{N_0} x^{N_1}\widetilde{g}^{(h)}(\mathbf{x}) = \int d\mathbf{k}\, e^{i\mathbf{k}\mathbf{x}}\hat{\partial}_0^{N_0}\partial_1^{N_1}\hat{g}^{(h)}(\mathbf{k}) \quad (165)$$

where $\hat{\partial}_0$ is the discrete derivative with respect to k_0, that is

$$\hat{\partial}_0 h(k_0) = \frac{\beta}{2\pi}\left(h\left(k_0 + \frac{2\pi}{\beta}\right) - h(k_0)\right).$$

We thus need an estimate for $\hat{\partial}_0^{N_0}\partial_1^{N_1}\hat{g}^{(h)}(\mathbf{k})$. To this end, we observe that

$$\hat{\partial}_0^{N_0}\partial_1^{N_1}\hat{g}^{(h)}(\mathbf{k})$$
$$= \sum_{P_1=1}^{N_1}\sum_{\sum_{i=1}^{N_1-P_1}p_i = N_1-P_1} A_{P_1,\{p_i\}}\hat{\partial}_0^{N_0}\partial_{\cos(k)}^{P_1}\hat{g}^{(h)}(\mathbf{k})\prod_{i=i}^{P_1}\frac{d^{p_i}}{dk^{p_i}}\sin(k) \tag{166}$$

where $A_{P_1,\{p_i\}}$ are combinatoric coefficients taking into account how many times the term $\partial_{\cos(k)}^{P_1}\hat{g}^{(h)}(\mathbf{k})\prod_{i=i}^{P_1}\frac{d^{p_i}}{dk^{p_i}}\sin(k)$ appears in $\partial_1^{N_1}\hat{g}^{(h)}(\mathbf{k})$. In particular

$$A_{P_1,\{p_i\}} \le \frac{N_1!}{P_1!}\prod_{i=1}^{N_1-P_1}\frac{1}{p_i!}$$

so that

$$\sum_{P_1=1}^{N_1}\sum_{\sum_i p_i = N_1-P_1} A_{P_1,\{p_i\}} \le (1+N_1)^{N_1}.$$

Observe now that,

$$\partial_{\cos(k)}\hat{g}^{(h)}(\mathbf{k}) = \frac{\partial_{\cos(k)}f_h(\mathbf{k})}{-ik_0(1+z_{h-1}) + (1+\alpha_{h-1})(\cos k - 1) + r + \gamma^{h^*}\mu_{h-1}}$$
$$- \frac{(1+\alpha_{h-1})f_h(\mathbf{k})}{(-ik_0(1+z_{h-1}) + (1+\alpha_{h-1})(\cos k - 1) + r + \gamma^{h^*}\mu_{h-1})^2}. \tag{167}$$

In turn we see from (57) that

$$|\partial_{\cos(k)}f_h(\mathbf{k})| \le a_0^{-1}\gamma^{-h}(1 + K|\lambda|)(1+\gamma)\|\chi_0'\|_\infty \le 36\|\chi_0'\|_\infty \gamma^{-h} \tag{168}$$

where $\|\chi_0'\|_\infty = \sup_p |\chi_0'(p)|$ while the second line of (167) can be bounded by $(1+K|\lambda|)a_0\gamma^{h+1} \le 3/8\gamma^h$. All togeher we get

$$\left|\partial_{\cos(k)}\hat{g}^{(h)}(\mathbf{k})\right| \le C'\gamma^{-2h}.$$

for a suitable C'. Using that $\hat{\partial}_0 h(k_0) = \partial_{k_0} h\left(k_0 + \theta \frac{2\pi}{\beta}\right)$ for a suitable $0 \le \theta \le 1$ we get, with a similar argument, that

$$\left|\hat{\partial}_0 \hat{g}^{(h)}(\mathbf{k})\right| \le C' \gamma^{-2h}.$$

Iterating these estimates gives

$$\left|\hat{\partial}_0^{N_0} \partial_{\cos(k)}^{P_1} \hat{g}^{(h)}(\mathbf{k})\right| \le C'_{N_0, P_1} \gamma^{-h(1+N_0+P_1)}. \tag{169}$$

It remains to estimate $\prod_{i=i}^{P_1} \frac{\mathrm{d}^{p_i}}{\mathrm{d}k^{p_i}} \sin(k)$. If $P_1 \le N_1/2$ we can use

$$\left|\prod_{i=i}^{P_1} \frac{\mathrm{d}^{p_1}}{\mathrm{d}k^{p_i}} \sin(k)\right| \le 1 \qquad \text{and} \qquad \gamma^{-h(1+N_0+P_1)} \le \gamma^{-h\left(1+N_0+\frac{N_1}{2}\right)} \tag{170}$$

while, if $P_1 > N_1/2$, at least $2P_1 - N_1$ of the p_i in the above product must be zero so that

$$\left|\prod_{i=i}^{P_1} \frac{\mathrm{d}^{p_1}}{\mathrm{d}k^{p_i}} \sin(k)\right| \le (2\sqrt{3})^{(2P_1 - N_1)} \gamma^{(2P_1 - N_1)\frac{h}{2}}$$

$$\gamma^{-h(1+N_0+P_1)} \gamma^{\frac{h}{2}(2P_1 - N_1)} = \gamma^{-h\left(1+N_0+\frac{N_1}{2}\right)}. \tag{171}$$

In both cases we get

$$|\hat{\partial}_0^{N_0} \partial_1^{N_1} \hat{g}^{(h)}(\mathbf{k})| \le \left((1+N_1)^{N_1} (2\sqrt{3})^{N_1} \sup_{P_1 \le N_1} C'_{N_0, P_1}\right) \gamma^{h\left(1+N_0+\frac{N_1}{2}\right)}. \tag{172}$$

It is clear that $\partial_0^{N_0} \partial_1^{N_1} \hat{g}^{(h)}(\mathbf{k})$ has the same support of $f_h(\mathbf{k})$ so that from (163) we get (164). Finally observe that

$$\widetilde{\partial}_0^{n_0} \partial_1^{n_1} \widetilde{g}^{(h)}(\mathbf{x}) = (i)^{n_0 + n_1} \int \mathrm{d}\mathbf{k} e^{i\mathbf{k}\mathbf{x}} k_0^{n_0} \sin^{n_1} k \hat{g}^{(h)}(\mathbf{k}). \tag{173}$$

The Lemma follows easily reasoning as above and using (161) for the extra powers of k_0 and k. $\qquad \square$

Proof of Lemma 3.1. As before, on the support of $f_h(\mathbf{k})$ we get

$$(1 + z_{h^*})|k_0| \le \gamma^{h+1} a_0 \qquad |(1 + \alpha_{h^*})(\cos k - \cos p_F)| \le \gamma^{h+1} a_0. \tag{174}$$

Writing

$$\cos k - \cos p_F = \cos p_F (\cos(k - \omega p_F) - 1) + \omega v_F \sin(k - \omega p_F)$$

and using (174) it easily follows that

$$|\sin(k - p_F)| \le C\gamma^h r^{-\frac{1}{2}}$$

and thus $|k - p_F| \le C\gamma^h r^{-\frac{1}{2}}$, so that

$$\int \chi_{\le h, \omega}(\mathbf{k}) \mathrm{d}\mathbf{k} \le C r^{-\frac{1}{2}} \gamma^{2h}. \tag{175}$$

Reasoning like in the proof of Lemma 2.2 we find that we need to show

$$|x_0^{N_0} x^{N_1} \widetilde{g}_\omega^{(h)}(\mathbf{x})| \le E\gamma^{h(1-N_0-N_1)} r^{\frac{N_1-1}{2}}. \tag{176}$$

To prove such an estimate we can closely follow the proof of Lemma 2.2. We first observe that (166) and (169) remain true. Indeed the only difference arise in (168) due to the presence of $\widetilde{\vartheta}$ in $\widetilde{f}_{h,\omega}$. Thus, for $h \leq h^*$ we get

$$|\partial_{\cos(k)} f_h(\mathbf{k})| \leq \gamma^{-h} \left(a_0^{-1}(1 + K|\lambda|)(1 + \gamma)\|\chi_0'\|_\infty + \frac{\gamma^h}{p_F}\|\widetilde{\vartheta}'\|_\infty \right) \leq C\gamma^{-h}. \tag{177}$$

Again we need to estimate $\prod_{i=i}^{P_1} \frac{\mathrm{d}^{p_i}}{\mathrm{d}k^{p_i}} \sin(k)$. For $P_1 \leq N_1/2$ we use (170) together with

$$\gamma^{-h(1+N_0+P_1)} \leq \gamma^{-h(1+N_0+N_1)} r^{N_1-P_1} \leq \gamma^{-h(1+N_0+N_1)} r^{\frac{N_1}{2}}$$

while for $P_1 > N_1/2$ we get

$$\left| \prod_{i=i}^{P_1} \frac{\mathrm{d}^{p_1}}{\mathrm{d}k^{p_i}} \sin(k) \right| \leq C\gamma^{(2P_1-N_1)h} r^{\frac{N_1}{2}-P_1}. \tag{178}$$

Observing that

$$\gamma^{-h(1+N_0+P_1)} \gamma^{(2P_1-N_1)h} r^{\frac{N_1}{2}-P_1} = \gamma^{-h(1+N_0+N_1)} r^{\frac{N_1}{2}} \left(\frac{\gamma^h}{r} \right)^{P_1}$$

$$\leq C\gamma^{-h(1+N_0+N_1)} r^{\frac{N_1}{2}}$$

and collecting we get

$$|\hat{\partial}_0^{N_0} \partial_1^{N_1} \hat{g}^{(h)}(\mathbf{k})| \leq \gamma^{-(1+N_0+N_1)h} r^{\frac{N_1}{2}} \tag{179}$$

The Lemma follows easily combining the above estimate with (175) and the analogous of (173). □

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