MATH 2401, PRACTICE TEST 3

1) Let Q be the triangle delimited by the lines $x = y$, $x = -y$ and $x = 1$. The triangle Q is occupied by a laminar object L with density $f(x, y) = ye^{-x}$. Compute the total mass M of the object L and its center of mass (x_M, y_M).

   The area in question can be described as

$$Q = \begin{cases} 0 \le x \le 1 \\ -x \le y \le x \end{cases}$$

so that we have:

$$M = \int_0^1 \int_{-x}^{x} ye^{-x} dydx = \int_0^1 e^{-x} \int_{-x}^{x} ydydx = 0$$

(I apologize: I did not realize this rather unphysical answer)

$$Mx_M = \int_0^1 \int_{-x}^{x} yxe^{-x} dydx = \int_0^1 xe^{-x} \int_{-x}^{x} ydydx = 0$$

$$My_M = \int_0^1 \int_{-x}^{x} y^2 e^{-x} dydx = \int_0^1 e^{-x} \int_{-x}^{x} y^2 dydx = \frac{2}{3} \int_0^1 e^{-x} x^3 = 4 - \frac{32}{3e}$$

2) Compute the integral of the function

$$f(x, y) = \operatorname{atan}\left(\frac{y}{x}\right)(x^2 + y^2)$$

   on the domain Q_1 given by the points (x, y) such that $-y \le x \le y$ and $x^2 + y^2 \le 1$.

   Using polar coordinates we get

$$Q = \begin{cases} 0 \le r \le 1 \\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \end{cases}$$

Moreover $\operatorname{atan}\left(\frac{y}{x}\right) = \theta$ and $x^2 + y^2 = r^2$ so that the intgral is

$$\int_0^1 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \theta r^2 r d\theta dr = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \theta d\theta \int_0^1 r^3 dr = \frac{\pi}{2} \frac{1}{4}$$

3) Let $f(x, y, z)$ be a continuous function and Q_2 the domain the domain bounded by the surfaces $z = 0$, $z = y$ and $x^2 = 1 - y$.
   a) Express

$$\int\int\int_V f(x, y, z) dxdydz$$

as an iterated integral. How many different way you have to do it?

$$Q_2 = \begin{cases} -1 \leq x \leq 1 \\ 0 \leq y \leq 1 - x^2 \\ 0 \leq z \leq y \end{cases}$$

so that

$$\int\int\int_V f(x, y, z)dxdydz = \int_{-1}^{1} \int_0^{1-x^2} \int_0^y f(x, y, z)dzdydx$$

There are 6 different way to write the integral as an iterated integral.

b) Let

$$f(x, y, z) = z^3 \frac{1}{\sqrt{1 + x^2}}.$$

Can you reduce the above iterated integral to an integral on one variable?

$$\int_{-1}^{1} \int_0^{1-x^2} \int_0^y z^3 \frac{1}{\sqrt{1+x^2}} dzdydx = \int_{-1}^{1} \frac{1}{\sqrt{1+x^2}} \int_0^{1-x^2} \int_0^y z^3 dzdydx =$$
$$\int_{-1}^{1} \frac{1}{\sqrt{1+x^2}} \int_0^{1-x^2} \frac{y^4}{4} dydx = \frac{1}{20} \int_{-1}^{1} \frac{1}{\sqrt{1+x^2}} (1 - x^2)^5 dx$$

4) Let Q_3 be an object delimited by the surfaces $z = y$, $z = -y$, $z = 1$, $x = 1$ and $x = -1$ and with a density $f(x, y, z) = x^2 y e^{-z}$. Compute the total mass M of Q_3 and its center of mass (x_M, y_M, z_M).

The region can be described as:

$$Q_3 = \begin{cases} 0 \leq z \leq 1 \\ -z \leq y \leq z \\ -1 \leq x \leq 1 \end{cases}$$

So that

$$M = \int_0^1 \int_{-z}^{z} \int_{-1}^{1} x^2 y e^{-z} dxdydz = \frac{2}{3} \int_{-1}^{1} \int_{-z}^{z} y e^{-z} dydz = 0$$

while

$$M x_M = \int_0^1 \int_{-z}^{z} \int_{-1}^{1} x^3 y e^{-z} dxdydz = \frac{1}{2} \int_{-1}^{1} \int_{-z}^{z} y e^{-z} dydz = 0$$

$$M y_M = \int_0^1 \int_{-z}^{z} \int_{-1}^{1} x^2 y^2 e^{-z} dxdydz = \frac{2}{3} \int_{-1}^{1} \int_{-z}^{z} y^2 e^{-z} dydz = \frac{4}{9} \int_{-1}^{1} z^3 e^{-z} = \frac{8}{3} - \frac{64}{9e}$$

5) Let Q_4 the region bounded by the surfaces $z = x^2 + y^2$ and $z = 1$. Compute the integral of

$$f_1(x, y, z) = (x^2 + y^2)e^{-z}$$
$$f_2(x, y, z) = x(x^2 + y^2)e^{-z}$$
$$f_3(x, y, z) = z(x^2 + y^2)e^{-z}$$

on Q_4.(Hint: use cylindrical coordinates)

Using cylindrical coordinates we get:

$$Q_4 = \begin{cases} 0 \le z \le 1 \\ 0 \le r \le z \\ 0 \le \theta \le 2\pi \end{cases}$$

so that the first integral is

$$\int_0^1 \int_0^z \int_0^{2\pi} r^2 e^{-z} r d\theta dr dz = 2\pi \int_0^1 \int_0^z r^3 e^{-z} dr dz = \frac{\pi}{2} \int_0^1 z^4 e^{-z} dz = \pi \left(12 - \frac{65}{2e}\right)$$

while

$$\int_0^1 \int_0^z \int_0^{2\pi} r \cos\theta r^2 e^{-z} r d\theta dr dz = 0$$

and

$$\int_0^1 \int_0^z \int_0^{2\pi} r^2 z e^{-z} r d\theta dr dz = \frac{\pi}{2} \int_0^1 z^5 e^{-z} dz = \pi \left(20 - \frac{163}{e}\right)$$