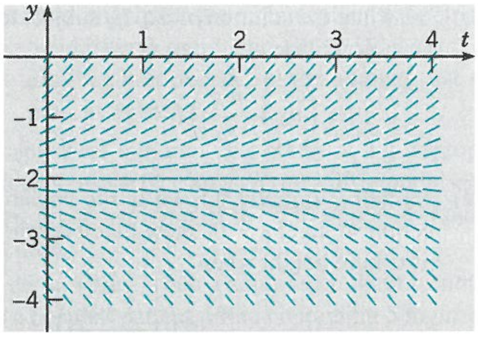

FIGURE 1.1.11 Direction field for Problem 18.

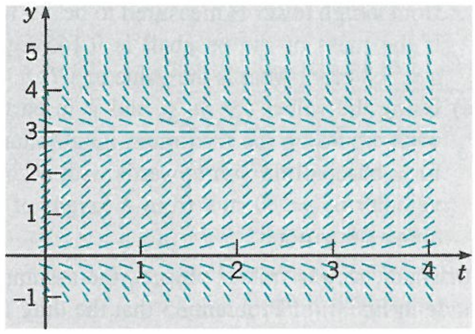

FIGURE 1.1.12 Direction field for Problem 19.

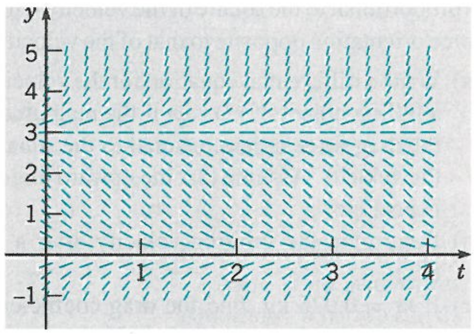

FIGURE 1.1.13 Direction field for Problem 20.

In each of Problems 21 through 28 draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that the right sides of these equations depend on t as well as y.

21. $y' = -2 + t - y$

22. $y' = te^{-2t} - 2y$

23. $y' = e^{-t} + y$

24. $y' = t + 2y$

25. $y' = 3\sin t + 1 + y$

26. $y' = 2t - 1 - y^2$

27. $y' = -(2t + y)/2y$

28. $y' = y^3/6 - y - t^2/3$

29. We get a discrete time approximation to the initial value problem

$$u' = -k(u - T_0), \quad u(0) = u_0 \tag{i}$$

on the time grid

$$0 = t_0, t_1, \ldots, t_n = t, \qquad t_j = j\Delta t,$$
$$j = 0, \ldots, n, \quad \text{where} \quad \Delta t = t/n$$

by approximating $u'(t_j)$ in the equation by the difference quotient

$$u'(t_j) \cong \frac{u(t_j) - u(t_j - \Delta t)}{\Delta t} = \frac{u(t_j) - u(t_{j-1})}{\Delta t} \tag{ii}$$

for each $j = 1, \ldots, n$.

(a) Show that replacing the derivative in Eq. (i) by the approximation in expression (ii) gives the difference equation

$$u(t_j) = (1 - k\Delta t)u(t_{j-1}) + kT_0\Delta t, \quad j = 1, \ldots, n. \tag{iii}$$

(b) Equation (iii) can be solved by iteration,

$$\begin{aligned} u(t_1) &= (1 - k\Delta t)u(t_0) + kT_0\Delta t \\ &= (1 - k\Delta t)u_0 + kT_0\Delta t, \\ u(t_2) &= (1 - k\Delta t)u(t_1) + kT_0\Delta t \\ &= (1 - k\Delta t)^2u_0 + (1 - k\Delta t)kT_0\Delta t \\ &\quad + kT_0\Delta t, \end{aligned}$$

and so forth. Show that successive iteration yields the result

$$u(t_n) = (1 - k\Delta t)^n u_0 + kT_0\Delta t\sum_{j=0}^{n-1}(1 - k\Delta t)^j,$$

or, using the formula for the partial sum of a geometric series,

$$u(t_n) = (1 - k\Delta t)^n u_0 + T_0\left[1 - (1 - k\Delta t)^n\right].$$

(c) Show that $\lim_{n\to\infty}(1 - kt/n)^n = e^{-kt}$. Then use this result to show that

$$\lim_{n\to\infty} u(t_n) = e^{-kt}(u_0 - T_0) + T_0,$$

the solution to the initial value problem (i).

30. Verify that the function in Eq. (23) is a solution of Eq. (22).