No books or notes allowed. No laptop or wireless devices allowed. Write clearly.

Name: ______________________________

Question:	1	2	3	4	Total
Points:	30	20	20	30	100
Score:					

Question 1 .... *30 point*

The following numbers x_i, $i = 1, \ldots, 17$, represent a sample of size $n = 17$ from a given population.

6.46	6.33	6.93	7.12	7.26
11.12	7.03	6.96	5.48	5.35
6.49	7.87	6.02	6.75	5.67
4.01	5.82			

(a) (10 points) Compute the sample median and IQR.

Solution:

After ordering the data you obtain

4.01	5.35	5.48	5.67	5.82
6.02	6.33	6.46	6.49	6.75
6.93	6.96	7.03	7.12	7.26
7.87	11.12			

so that:

$$\tilde{x} = 6.49$$

$$1q = (5.67 + 5.82)/2 = 5.75 \qquad 3q = (7.03 + 7.12)/2 = 7.08$$

$$IQR = 7.08 - 5.75 = 1.33$$

$$lf = 5.82 \qquad uf = 7.03$$

$$fs = 7.03 - 5.82 = 1.21$$

(b) (10 points) Knowing that $\sum_{i=1}^{17} x_i = 112.67$ and $\sum_{i=1}^{17} x_i^2 = 781.35$ compute the sample mean and standard deviation.

Solution:

$$\bar{x} = \frac{112.67}{17} = 6.63$$

$$s^2 = \frac{1}{16}\left(781.35 - \frac{112.67^2}{17}\right) = 2.16$$

(c) (10 points) After checking for outlier, draw a box plot of the data.

Solution: The inner fences are: $5.75 - 1.5 \cdot 1.33 = 3.755$ and $7.08 - 1.5 \cdot 1.33 = 9.08$ while the outer fences are $5.75 - 3 \cdot 1.33 = 1.76$ and $7.08 - 3 \cdot 1.33 = 11.07$. Clearly there are no suspected outliers and 11.12 is the only outlier.

Question 2 . . . . *20 point*

Let $\mathcal{S}$ be a sample space and $A, B, C \subset \mathcal{S}$ be three mutually independent events.

(a) (10 points) Show that A' and B are independent.(**Hint:** use that $B = (A \cap B) \cup (A' \cap B)$)

Solution:
You need to show that

$$P(A' \cap B) = P(A')P(B)$$

Since $(A \cap B) \cup (A' \cap B) = B$ and $(A \cap B) \cap (A' \cap B) = \emptyset$ we have

$$P(A \cap B) + P(A' \cap B) = P(B)$$

or

$$P(A' \cap B) = P(B) - P(A \cap B)$$

But $P(A \cap B) = P(A)P(B)$ so that we get

$$P(A' \cap B) = P(B) - P(A)P(B) = P(B)(1 - P(A)) = P(A')P(B).$$

(b) (10 points) Show that A and $B \cup C$ are independent.(**Hint:** use that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$)

Solution: You need to show that

$$P(A \cap (B \cup C)) = P(A)P(B \cup C)$$

We have

$$P(A \cap (B \cup C)) = P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))$$

But $(A \cap B) \cap (A \cap C) = A \cap B \cap C$ so and $P(A \cap B \cap C) = P(A)P(B)P(C)$ so that

$$\begin{aligned} P(A \cap (B \cup C)) &= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C) = \\ &= P(A)(P(B) + P(C) - P(B)P(C)) \end{aligned}$$

On the other hand, we have

$$P(B \cup C) = P(B) + P(C) - P(B \cap C) = P(B) + P(C) - P(B)P(C)$$

so that

$$P(A \cap (B \cup C)) = P(A)P(B \cup C)$$

Question 3 . . . . *20 point*

Let X_1 and X_2 be two r.v that can take the values -1, 0 and 1. Assume that each of the three value has the same probability for both X_1 and X_2 and that the events $\{X_1 = i\}$ and $\{X_2 = j\}$ are independent for every i and j, $i, j = -1, 0, 1$.

(a) (10 points) Compute the p.m.f. of $Y = X_1 + X_2$ and $Z = X_1 X_2$.

Solution: Observe that Y can take the values $-2, -1, 0, 1, 2$. There is one outcome each for which $Y = -2$ or $Y = 2$, two outcomes each for $Y = -1$ or $Y = 1$ and three outcomes for $Y = 0$ so that:

$$p_Y(-2) = p_Y(2) = \frac{1}{9} \qquad p_Y(-1) = p_Y(1) = \frac{2}{9} \qquad p_Y(0) = \frac{1}{3}$$

On the other hand, Z can take the values $-1, 0, 1$. Two outcomes give -1 or 1 and five outcomes give 0. Thus

$$p_Z(-1) = p_Z(1) = \frac{2}{9} \qquad p_Z(0) = \frac{5}{9}$$

(b) (10 points) Compute $E(Y)$, $E(Z)$ and $V(Y)$, $V(Z)$.

Solution:

$$E(Y) = -2 \cdot p_Y(-2) - 1 \cdot p_Y(-1) + 0 \cdot p_Y(0) + 1 \cdot p_Y(1) + 2 \cdot p_Y(2) = 0$$
$$E(Z) = -1 \cdot p_Z(-1) + 0 \cdot p_Z(0) + 1 \cdot p_Z(1) = 0$$

While

$$E(Y^2) = 4 \cdot p_Y(-2) + 1 \cdot p_Y(-1) + 0 \cdot p_Y(0) + 1 \cdot p_Y(1) + 4 \cdot p_Y(2) = \frac{4}{3}$$
$$E(Z^2) = 1 \cdot p_Z(-1) + 0 \cdot p_Z(0) + 1 \cdot p_Z(1) = \frac{4}{9}$$

so that

$$V(Y) = E(Y^2) - E(Y)^2 = \frac{4}{3}$$
$$V(Z) = E(Z^2) - E(Z)^2 = \frac{4}{9}$$

Question 4 . . . . *30 point*

A factory produces 1000 computers. Each computer has a probability $p = 0.003$ to have an internal defect. The factory has a quality control department that is able to detect and discard a defective computer with probability 1. It also has a probability $s = 0.002$ of discarding a working computer.

(a) (10 points) Let X be the number of defected computer among the 1000 produced. Write the p.m.f. of X and compute $E(X)$ and σ_X.

Solution: Clearly X is a Binomial r.v. with parameter 1000 and 0.003. It follows that

$$p_x(x) = P(X = x) = \binom{1000}{x} 0.003^x 0.997^{1000-x}$$

Moreover

$$E(X) = 1000 \cdot 0.003 = 3 \qquad \sigma_X = \sqrt{1000 \cdot 0.003 \cdot 0.99} = \sqrt{2.97} = 1.72$$

(b) (10 points) Compute the probability that a randomly selected computer will be discarded by the quality control department and the probability that a discarded computer is actually defective.

Solution:

Call A the event $\{\textit{computer is defected}\}$ and B the event $\{\textit{computer is discarded}\}$. Then

$$P(B) = P(B|A)P(A) + P(B|A')P(A') = 1 \cdot 0.003 + 0.002 \cdot 0.997 = 0.005$$

while

$$P(A|B) = P(B|A)\frac{P(A)}{P(B)} = 1\frac{0.003}{0.005} = 0.6$$

(c) (10 points) Call Y the r.v. that describes number of discarded computers that are working. Write the p.m.f. of Y. Using the Poisson approxiamtion, compute the probability that $Y = 4$.

Solution:

Clearly Y is a Binomial r.v.. with parameters 1000 and p, where is the probability that a computer is both working and discarded. This last probability is the probability that the computer is working time the probability that it will be discarded given that it is working. Thus $p = 0.997 * 0.002 = 0.002$.

It follows that

$$p_Y(y) = \binom{1000}{y} 0.998^{1000-y} 0.002^y.$$

Since $1000 * 0.002 = 2$ we have

$$P(Y = 2) \simeq e^{-2} \frac{2^4}{4!} = 0.09.$$