MATH 3235, Test I **Houdré**
Thursday October 11, 2018 **Name:** Solutions

Answer all questions; show all work; closed books, no calculators. THE HONOR CODE APPLIES TO THIS CLASS

Problem	Points	Score
1	50	50
2	50	50
3	50	50
4	50	50
Bonus	20	20
Total	200	220

1. (a) (25pts) Let A and B be two events which occur with respective probability $\mathbb{P}(A) = 1/4$, $\mathbb{P}(B) = 1/2$ and such that $\mathbb{P}(A \cup B) = 3/5$. Are A and B independent events? Justify your answer.

Since $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$

We have

$$\mathbb{P}(A \cap B) = \frac{1}{4} + \frac{1}{2} - \frac{3}{5} = \frac{3}{4} - \frac{3}{5} = \frac{3}{20}$$

But

$$\mathbb{P}(A)\mathbb{P}(B) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

Since

$\mathbb{P}(A \cap B) \neq \mathbb{P}(A)\mathbb{P}(B)$, the

two events are not $\perp\!\!\!\perp$

they are dependent.

independent

(b) (25pts) Let A, B and C be three events which occur with respective probability $\mathbb{P}(A) = 1/3$, $\mathbb{P}(B) = 1/2$ and $\mathbb{P}(C) = 1/5$. Find the probability that exactly two of the events occur.

Exactly two corresponds to

$$(A^c \cap B \cap C) \cup (A \cap B^c \cap C) \cup (A \cap B \cap C^c)$$

pairwise disjoint

Hence

$$\mathbb{P}(\text{exactly two}) = \mathbb{P}(A^c \cap B \cap C) + \mathbb{P}(A \cap B^c \cap C) + \mathbb{P}(A \cap B \cap C^c)$$

$$\stackrel{\perp\!\!\!\perp}{=} \mathbb{P}(A^c)\mathbb{P}(B)\mathbb{P}(C) + \mathbb{P}(A)\mathbb{P}(B^c)\mathbb{P}(C) + \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C^c).$$

$$= \frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{4}{5}$$

$$= \frac{2 + 1 + 4}{30} = \frac{7}{30}$$

five biased with $\mathbb{P}(H)=1/4$, $\mathbb{P}(T)=3/4$

2. (50pts) Consider two urns. The first one has two white balls and seven black balls and the second has seven white balls and ~~six~~ black balls. A ~~fair~~ coin is flipped and if heads comes up a ball is drawn from the first urn while if the coin shows tails then a ball is drawn from the second urn. Given that a white ball has been drawn, what is the probability that the outcome of your coin toss was head?

$W=$ a white ball has been drawn

$H=$ the toss was heads, $T=$ the toss was tails

$$P(H|W)=\frac{\mathbb{P}(H\cap W)}{\mathbb{P}(W)}=\frac{\mathbb{P}(W|H)\,\mathbb{P}(H)}{\mathbb{P}(W\cap H)+\mathbb{P}(W\cap T)}$$

$$=\frac{\mathbb{P}(W|H)\,\mathbb{P}(H)}{\mathbb{P}(W|H)\,\mathbb{P}(H)+\mathbb{P}(W|T)\,\mathbb{P}(T)}$$

$$=\frac{2/9\cdot 1/4}{2/9\cdot 1/4+7/12\cdot 3/4}=\frac{\frac{1}{18}}{\frac{1}{18}+\frac{7}{12}\cdot\frac{3}{4}}$$

$$=\frac{1}{18}\left[\frac{1}{\frac{1}{18}+\frac{7}{16}}\right]=\frac{1}{\frac{18}{2}}\left[\frac{1}{\frac{8+63}{72}}\right]$$

$$=\frac{1}{9}\left(\frac{72}{71}\right)$$

$$=8/71$$

(a)

3. (50pts) I throw ~~two~~ three sided fair die, and a fair coin, and the die and the coin are $\perp\!\!\!\perp$ and record their respective score S_1 and S_2. Let X be the sum of the scores, i.e., $X = S_1 + S_2$ and let Y be the quotient of the scores, i.e., $Y = S_2/S_1$. Find the joint probability mass function of X and Y. Are X and Y independent?

The score S_1 is the # of dots on the top face

The score S_2 is 0 if heads and 1 if tails.

X takes the values 1, 2, 3, 4

Y " " 0, 1, 1/2, 1/3.

X \ Y	0	1/3	1/2	1
1	1/6	0	0	0
2	1/6	0	0	1/6
3	1/6	0	1/6	0
4	0	1/6	0	0

Not $\underline{\perp\!\!\!\perp}$ since

$$\mathbb{P}(X=1, Y=1) = 0 \neq \mathbb{P}(X=1)\,\mathbb{P}(Y=1)$$

$$= \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

Note: This problem is similar But also different from a problem from the old test!!

4. (50pts) Each morning my grandmother used to go to her garden to collect the eggs laid over the previous night by her hens, in order to sell them to the weekly market. Unfortunately, each time my grand-mother collected an egg there was a $1/100$ change she will break it (and thus the broken eggs could not be sold to the market) and all the breaks are independent of each other. Assume that the number of eggs laid during a week is a Poisson random variable N with parameter $\lambda = 100$. Denote by X the number of eggs my grandmother weekly sells at the market.

(a) (30pts) Fix $n \in \{0, 1, 2, \ldots\}$, and find $p_{X|N=n}$, the conditional pmf of X given $\{N = n\}$,

If $n=0$, then necessarily $X=0$, so $\mathbb{P}(X=0 | N=0) = 1$

If $n=1$, then $X=0$ or 1 with

$\mathbb{P}(X=0|N=1) =$ Probability that the single laid egg is broken $= 10^{-2}$

$\mathbb{P}(\cancel{N}=1|N=1) = 1-10^{-2}$

If $n=2$, then $X=0, 1$ or 2.

$\mathbb{P}(X=0|N=2) = \mathbb{P}$(both eggs get broken) $= (10^{-2})^2$

$\mathbb{P}(X=1|N=2) = \mathbb{P}$(only one of the eggs get broken) $= 2\,10^{-2}(1-10^{-2})$

$\mathbb{P}(X=2|N=2) = \mathbb{P}$(no eggs get broken) $= (1-10^{-2})^2 = \left(\frac{99}{100}\right)^2$

More generally, for $n \geq 3$, X takes the values

values $0, 1, 2, \ldots, n$ with

(b) (10pts) Find $\mathbb{E}(X|N = n)$. $\mathbb{P}(X = x \mid N = n) = \binom{n}{x}(1-10^{-2})^{x}(10^{-2})^{n-x}$

So for $n \geq 1$, $(X \mid N = n) \sim Bin(n, 1-10^{-2}) = Bin(n, \frac{99}{100})$

and $= 1$ for $n = 0$

Now $\mathbb{E}(X \mid N = n) = \sum_{x=0}^{n} x \mathbb{P}(X = x \mid N = n)$

$= n(1-10^{-2}) = \frac{99n}{100}$

and

$\mathbb{E}(X \mid N = 0) = 0 \, \mathbb{P}(X = 0 \mid N = 0) = 0$

So $\mathbb{E}(X \mid N) = \frac{99}{100} N$, where $N \sim \mathcal{P}(10^2)$

(c) (10pts) Find $\mathbb{E}X$, the expectation of X.

$\mathbb{E}X = \mathbb{E}(\mathbb{E}(X \mid N)) = (1-10^{-2})\mathbb{E}N = \frac{99}{100} \cdot 10^2$

$= 99$

or write as

$$\mathbb{E}X = \sum_{n=0}^{\infty} \mathbb{E}(X \mid N = n)\mathbb{P}(N = n)$$

$$= \sum_{n=0}^{\infty} n \cdot \frac{99}{100} e^{-100} \frac{(100)^n}{n!} = e^{-100}\, 99 \sum_{n=0}^{\infty} n \frac{(100)^{n-1}}{n!}$$

$$= e^{-100}\, 99 \sum_{n=1}^{\infty} \frac{(100)^{n-1}}{(n-1)!} = e^{-100}\, 99 e^{100} = 99$$

(As expected!)

Sol 1

(d) (Bonus 20 points) Find G_X the probability generating function of X. What do you conclude about X? **Hint: To find G_X, remember the partition theorem.**

$$G_X(s) = \mathbb{E}(s^X) = \mathbb{E}(s^X \mid N=0)\mathbb{P}(N=0) + \mathbb{E}\sum_{n=1}^{\infty}(s^X \mid N=n)\mathbb{P}(N=n)$$

$$= s^0 \cdot e^{-\lambda} + \mathbb{E}\sum_{n=1}^{\infty}\left(s^{X_1+\dots+X_N} \mid N=n\right)\mathbb{P}(N=n)$$

$$= e^{-\lambda} + \mathbb{E}\sum_{n=1}^{\infty} s^{X_1+\dots+X_n}\mathbb{P}(N=n), \quad X_1,\dots,X_n \text{ iid } \mathrm{Ber}(p)$$

$\perp\!\!\!\perp$ of N and X

$$= e^{-\lambda} + \sum_{n=1}^{\infty}\mathbb{E}(s^{X_1+\dots+X_n})\frac{e^{-\lambda}\lambda^n}{n!}$$

$\perp\!\!\!\perp$ of X_i $(q = 1-p)$

$$= \sum_{n=0}^{\infty}(q+ps)^n\frac{e^{-\lambda}\lambda^n}{n!}, \quad (X_1+\dots+X_n \sim \mathrm{Bin}(n,p))$$

Note that

$$= e^{-\lambda}e^{\lambda(q+ps)} = e^{\lambda(q-1+ps)} = e^{\lambda p(s-1)}$$

where in our case, $\lambda = 100$, $p = \frac{99}{100}$, so $\lambda p = 99$

So we conclude that $X \sim \mathcal{P}(99)$.

<u>Sol 2</u>: $\mathbb{E}\, s^X = \sum_{x=0}^{\infty} s^x \,\mathbb{P}(X=x)$.

Now by the partition theorem, $\mathbb{P}(X=x) = \sum_{n=0}^{\infty} \mathbb{P}(X=x \mid N=n)\,\mathbb{P}(N=n)$

Now $\mathbb{P}(X=x \mid N=n)\,\mathbb{P}(N=n) = \binom{n}{x}\left(\frac{1}{100}\right)^{n-x}\left(\frac{99}{100}\right)^{x} e^{-100}\frac{(100)^n}{n!}$

for $n - x \geq 0$ $\quad = \binom{n}{x}\frac{1}{n!}(99)^x e^{-100}$

So $\mathbb{P}(X=x) = e^{-100}(99)^x \sum_{n=x}^{\infty} \frac{n!}{x!\,(n-x)!}\,\frac{1}{n!} = e^{-100}(99)^x \sum_{n=x}^{\infty}\frac{1}{x!\,(n-x)!}$

$= e^{-100}\frac{(99)^x}{x!}\sum_{n=0}^{\infty}\frac{1}{n!} = e^{-100}e^{1}(99)^x = e^{-99}(99)^x$

So $\mathbb{E}\, s^X = \sum_{x=0}^{\infty} s^x \frac{(99)^x}{x!} e^{-99} = e^{-99}\sum_{x=0}^{\infty}\frac{(99s)^x}{x!} = e^{-99}e^{99s}$

$= e^{99(s-1)}, \quad \forall s \in \mathbb{R}$.

So $X \sim \mathcal{P}(99)$.