MATH 3235, Test II
Tuesday April 10th, 2018

Name: Solutions

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Answer all questions; show all work; closed books, no calculators. THE HONOR CODE APPLIES

Problem	Points	Score
1	60	60
2	60	60
3	80	80
Total	200	200

1. A player is flipping a coin with for each flip a probability of success equal to $0 < p < 1$ and the flips are independent. A turn consists of a sequence of flips up to the first failure. Let n be a positive integer and let X be the total number of successes in n turns.

(40pts) Find G_X, the probability generating function of X.

First note that $X = X_1 + \cdots + X_n$, where X_i, $i=1,\dots,n$ is the # of successes during the ith turn. The X_i are $\perp\!\!\!\perp$ and identically distributed. So

$$G_X(s) = \prod_{s=1}^{n} G_{X_i}(s) = \left(G_{X_1}(s)\right)^n$$

Now X_1 takes the values $0, 1, 2, \dots$

$$\mathbb{P}(X_1 = k) = p^k q$$ (the first failure is on the $(k+1)$th flip).

$$\text{So } G_{X_1}(s) = \sum_{k=0}^{\infty} p^k q s^k = q \sum_{k=0}^{\infty} p^k s^k$$

$$= \frac{q}{1-ps}, \quad |ps| < 1.$$

and $$G_X(s) = \left(\frac{q}{1-ps}\right)^n$$

(b) (20pts) Find the mean and the variance of X

$$\mathbb{E}X = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathbb{E}X_i = n\,\mathbb{E}X_1 = n \cdot G'_{X_1}(1)$$

But $G'_{X_1}(s) = \dfrac{pq}{(1-ps)^2}$ and so

$$G'_{X_1}(1) = \frac{pq}{q^2} = \frac{p}{q}.$$

Hence $\mathbb{E}X = \dfrac{np}{q}$. (10pts)

$$\mathrm{Var}\,X = n\,\mathrm{Var}\,X_1 = n\left(\mathbb{E}(X_1(X_1-1)) - (\mathbb{E}X_1)^2 + \mathbb{E}X_1\right)$$

$$= n\left(G''_{X_1}(1) - \left(\frac{p}{q}\right)^2 + \frac{p}{q}\right).$$

But, $G''_{X_1}(s) = \dfrac{2p^2q}{(1-ps)^3}$ and so $G''_{X_1}(1) = \dfrac{2p^2q}{q^3} = \dfrac{2p^2}{q^2}$

So

$$\mathrm{Var}\,X = n\left(\frac{2p^2}{q^2} - \frac{p^2}{q^2} + \frac{p}{q}\right) = n\left(\frac{p^2}{q^2} + \frac{p}{q}\right) = \frac{np}{q}\left(\frac{p+q}{q}\right)$$

$$= \frac{np}{q^2}.$$

2. Let U be a uniform random variable on the interval $(0,1)$ and let $X = \frac{3U}{1-U}$.

(a) 4 pts Find the cumulative distribution function of X.

First note that $X > 0$ and so for $x \leq 0$, $F_X(x) = \mathbb{P}(X \leq x) = 0$. So let $x > 0$,

then $\mathbb{P}(X \leq x) = \mathbb{P}\left(\frac{3U}{1-U} \leq x\right) = \mathbb{P}(3U \leq x(1-U))$

$= \mathbb{P}((3+x)U \leq x) = \mathbb{P}\left(U \leq \frac{x}{3+x}\right) = \frac{x}{3+x}$

(b) (2 pts) Find the expectation of $\frac{X+3}{X+1}$.

$$f_X(x) = \begin{cases} F_X'(x) = \frac{(3+x)-x}{(3+x)^2} = \frac{3}{(3+x)^2}, & x > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

So $$\mathbb{E}\left(\frac{X+3}{X+1}\right) = \int_0^\infty \frac{(x+3)}{(x+1)} \cdot \frac{3}{(3+x)^2}\, dx = 3\int_0^\infty \frac{1}{(x+1)(x+3)}\, dx$$

$$= \frac{3}{2}\int_0^\infty \left(\frac{1}{x+1} - \frac{1}{x+3}\right) dx = \frac{3}{2}\lim_{N\to+\infty}\Big[\ln(1+x) - \ln(3+x)\Big]_0^N = \frac{3}{2}\lim_{N\to+\infty}\left[\ln\left(\frac{1+x}{3+x}\right)\right]_0^N = \frac{3}{2}\ln 3.$$

Other sol: $\frac{X+1}{X+3} = \frac{3}{2U+1}$, so $\mathbb{E}\left(\frac{X+1}{X+3}\right) = 3\int_0^1 \frac{1}{2u+1}\,du = \frac{3}{2}\ln(2u+1)\Big]_0^1 = \frac{3}{2}\ln 3.$

3. Let (X, Y) be a bivariate normal vector with pdf given by

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right\},$$

for some fixed $\rho \in (-1, 1)$. Let $Z = (Y - \rho X)/\sqrt{1-\rho^2}$.

(a) (20pts) Find $\mathbb{E}(YZ)$.

$$\mathbb{E}(YZ) = \mathbb{E}\left(\frac{Y^2 - \rho XY}{\sqrt{1-\rho^2}}\right) = \frac{1}{\sqrt{1-\rho^2}}(\mathbb{E}Y^2 - \rho\,\mathbb{E}XY)$$

$$= \frac{1}{\sqrt{1-\rho^2}}(1-\rho^2) = \sqrt{1-\rho^2}$$

Since as seen in class $X \sim N(0,1)$

$Y \sim N(0,1)$

and $\mathbb{E}XY = \rho$.

(b) (50 pts) Find the joint pdf of X and Z. What do you conclude?

$(X, Z) = (u(X,Y), v(X,Y))$ where $u(X,Y) = X$

and $v(X,Y) = (Y - \rho X)/\sqrt{1-\rho^2}$.

Then

$$f_{(X,Z)}(x,z) = f(x, y(x,z))\,|J(x,z)| \text{ where}$$

$$|J(x,z)| = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial z} \\ \partial y/\partial x & \partial y/\partial z \end{vmatrix}$$ and where $y(x,z) = \sqrt{1-\rho^2}\,z + \rho x$

So $|J(x,z)| = \begin{vmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{vmatrix} = \sqrt{1-\rho^2}$

So

$$f_{(X,Z)}(x,z) = \frac{\sqrt{1-\rho^2}}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho x\left(\sqrt{1-\rho^2}\, z + \rho x\right) + \left(\sqrt{1-\rho^2}\, z + \rho x\right)^2\right)\right)$$

$$= \frac{1}{2\pi} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2(1-\rho^2) + z^2(1-\rho^2)\right)\right)$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{2}} e^{-\frac{z^2}{2}}.$$ So X and Z are $\perp\!\!\!\perp$ standard normal r.v.

(c) (10pts) Using (b), or otherwise, find $\mathbb{E}(Z|X)$.

Z and X are $\perp\!\!\!\perp$ and so $\mathbb{E}(Z|X) = \mathbb{E}(Z)$
$= 0!$

or:

$$\mathbb{E}(Z|X) = \mathbb{E}\left(\frac{Y - \rho X}{\sqrt{1-\rho^2}} \,\middle|\, X\right)$$

$$= \frac{1}{1-\rho^2}\left[\mathbb{E}(Y|X) - \rho\,\mathbb{E}(X|X)\right]$$

$$= \frac{1}{1-\rho^2}\left(\rho X - \rho X\right) = 0$$