No books or notes allowed. No laptop or wireless devices allowed. Show all your work for full credit. **Write clearly and legibly**.

Name (print): ____________________

Question:	1	2	3	Total
Points:	75	15	30	120
Score:				

Question 1 . . . . *75 point*

Let X and Y be two jointly continuous r.v. with density function

$$f(x,y) = \begin{cases} e^{-y} & 0 < x < y \\ 0 & \text{otherwise.} \end{cases}$$

(a) (15 points) Find the marginal density functions $f_X(x)$ and $f_Y(y)$.

Solution: We have

$$f_X(x) = \int_x^\infty e^{-y} dy = e^{-x} \qquad \text{if } x > 0$$

while

$$f_Y(y) = \int_0^y e^{-y} dx = ye^{-y} \qquad \text{if } y > 0.$$

(b) (15 points) Find the conditional density functions $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

Solution: We have

$$f_{X|Y}(x|y) = \frac{e^{-y}}{ye^{-y}} = \frac{1}{y} \qquad \text{if } 0 < x < y$$

while

$$f_{Y|X}(y|x) = \frac{e^{-y}}{e^{-x}} = e^{-(y-x)} \qquad \text{if } 0 < x < y.$$

(c) (15 points) Find the expectation $\mathbb{E}(X|Y = y)$ of X given $Y = y$.

Solution: Since the density $f_{X|Y}(x|y)$ is uniform in $[0, y]$ we get that

$$\mathbb{E}(X|Y) = \frac{y}{2}$$

(d) (15 points) Let now $U = X$ and $V = Y - X$. Find the joint density function $F_{U,V}(u, v)$ of U and V.

Solution: Inverting we get $X = U$ and $Y = U + V$. Since the Jacobian of the change of variable is 1 we get

$$f_{U,V}(u, v) = \begin{cases} e^{-(u+v)} & u, v > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(e) (15 points) Are X and Y independent? What about U and V?

Solution: Since $f_{X|Y}(x|y)$ depends explicitly on y, X and Y are not independent.

Clearly we have

$$f_{U,V}(u,v) = f_U(u) f_V(v)$$

where

$$f_U(u) = e^{-u}$$
$$f_V(v) = e^{-v}$$

so that U and V are independent.

Question 2 . . . . *15 point*

Given a continuous r.v. X let $\tilde{x}_{\frac{1}{2}}$, the *median*, be defined as

$$\mathbb{P}\left(X \leq \tilde{x}_{\frac{1}{2}}\right) = \frac{1}{2}$$

and $\tilde{x}_{\frac{3}{4}}$, the *(upper)–quartile*, be defined as

$$\mathbb{P}\left(X \leq \tilde{x}_{\frac{3}{4}}\right) = \frac{3}{4}.$$

Assume now that X is an exponential r.v. Show that

$$\tilde{x}_{\frac{3}{4}} = 2\tilde{x}_{\frac{1}{2}}.$$

Solution: We have

$$F_X(x) = 1 - e^{-\lambda x}$$

so that $\tilde{x}_{\frac{1}{2}}$ satisfies

$$e^{-\lambda \tilde{x}_{\frac{1}{2}}} = \frac{1}{2}$$

and similarly $\tilde{x}_{\frac{3}{4}}$ satisfies

$$e^{-\lambda \tilde{x}_{\frac{3}{4}}} = \frac{1}{4}.$$

Thus

$$e^{-\lambda 2\tilde{x}_{\frac{1}{2}}} = \left(e^{-\lambda \tilde{x}_{\frac{1}{2}}}\right)^2 = e^{-\lambda \tilde{x}_{\frac{3}{4}}}.$$

Question 3 . . . . *30 point*

You are waiting for the bus at a bus station. To go home you can take two bus lines: line A or line B. The waiting time for the next line A bus is described by an exponential r.v. T_A with parameter 2 while the waiting time for the next line B bus is described by an exponential r.v. T_B with parameter 3. Finally T_A and T_B are independent.

(a) (15 points) Find the distribution of the waiting time T till the arrival of the next useful bus. (**Hint**: Observe that $T = \min\{T_A, T_B\}$. Compute $\mathbb{P}(T > t)$.)

Solution: Clearly we have

$$T = \min\{T_A, T_B\}$$

so that

$$\begin{aligned}\mathbb{P}(T > t) =& \mathbb{P}(\min\{T_A, T_B\} > t) = \mathbb{P}(T_A > t \ \& \ T_B > t) = \\ & \mathbb{P}(T_A > t)\mathbb{P}(T_B > t) = e^{-5t}\end{aligned}$$

Thus

$$F(t) = \mathbb{P}(T < t) = 1 - e^{-5t}$$

so that T is an exponential with parameter 5.

(b) (15 points) The first bus that passes is full and you cannot take it. How long will you have to wait, in average, till the next bus arrives? Justify your answer.

Solution: *Natural interpretation.* Assume that the first bus arriving is from line A. Then another bus from line A will arrive in a time T_A with exponential distribution with parameter 2. The waiting time for the arrival of a bus from line B is still an exponential distribution with parameter 3 due to the loss of memory property of the exponential distribution. Thus the waiting time is again exponentially distributed with parameter 5 and the average waiting time is 1/5. The same result holds if the first bus arriving is from line B. Thus the average waiting time is 1/5.

Possible different interpretation. If you think there is only one bus for each line than the waiting time will be, again due to loss of memory,

$$\frac{1}{3}\mathbb{P}(T_A < T_B) + \frac{1}{2}\mathbb{P}(T_B < T_A).$$

We have

$$\mathbb{P}(T_A < T_B) = 6\int_{t_A<t_B} e^{-2t_A-3t_B}dt_A\,dt_B =$$
$$6\int_0^\infty dt_A\int_{t_A}^\infty dt_B e^{-2t_A-3t_B} = 2\int_0^\infty e^{-5t_A}dt_A = \frac{2}{5}$$

so that

$$\mathbb{P}(T_B < T_A) = \frac{3}{5}$$

and the average waiting time is 13/30.

Useful Formulas

- **Exponential Distribution**: if T is an exponential r.v. with parameter λ then its density function is
$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if} \quad t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
while $E(T) = 1/\lambda$ and $F(x) = P(X \leq x) = 1 - e^{-\lambda x}$.