Exercises

1.10) We can write

$$A \Delta B = (A \cup B) \cap (\Omega \setminus A \cap B)$$

If A and B are events Than

$A \cup B$ and $A \cap B$ are events. ~~and~~

So $\Omega \setminus A \cap B$ is an event and

finally $A \Delta B$ is an event.

1.17) We should check (a), (b) and (c) in The definition.

(a) evident

(b) $\mathbb{P}(\phi) = 0$ is evident, $\mathbb{P}(\Omega) = 1$ follows

(c) ~~If~~ Let $A = \bigcup_i A_i$ ~~and~~ with $A_i \cap A_j = \phi$.

If $\omega \in A$ Then ~~as~~ There exists

one and only one A_j such That

$\omega \in A_j$. Thus

$$Q(A) = \sum_{i:\omega_i \in A} p_i = \sum_j \sum_{i:\omega_i \in A_j} p_i = \sum_j Q(A_j)$$

In particular we never used that $\mathcal{F}$ is the power set of Ω. For example we can take

$$\mathcal{F} = \{\emptyset, \Omega\}$$

and Q would a probability measure on $\mathcal{F}$.

1.21) The event needed is

$$D = \big((A\cap B) \cup (A \cap C) \cup (A \cap D)\big) \setminus (A \cap B \cap C)$$

$$= D_1 \cup D_2 \cup D_3 \quad \text{where}$$

$$D_1 = (A \cap B) \setminus (A \cap B \cap C)$$

$$D_2 = (A \cap C) \setminus (A \cap B \cap C)$$

$$D_2 = (B \cap C) \setminus (A \cap B \cap C)$$

Observe that $D_i \cap D_j = \emptyset$ if $i \neq j$.

Moreover

$$\mathbb{P}(D_1) = \mathbb{P}(A \cap B) - \mathbb{P}(A \cap B \cap C)$$

since $A \cap B \cap C \subset A \cap B$. Similarly

$$\mathbb{P}(D_2) = \mathbb{P}(A \cap C) - \mathbb{P}(A \cap B \cap C)$$

$$\mathbb{P}(D_3) = \mathbb{P}(B \cap C) - \mathbb{P}(A \cap B \cap C)$$

so that

$$\mathbb{P}(D) = \mathbb{P}(A \cap B) + \mathbb{P}(B \cap C) + \mathbb{P}(A \cap C) - 3 \cdot \mathbb{P}(A \cap B \cap C) = \frac{6}{10} = \frac{3}{5}$$

1.27) The total number of possible bridge hands is

$$\frac{52!}{13!^4}$$

On the other hand there are

$4! = 24$ way to give an ace to

each player and

$$\frac{48!}{12!^4}$$

way to distribute the remaining 48 cards to the 4 players, 12 each.

Thus the probability is

$$4!\,\frac{48!}{12!^4}\Big/\frac{52!}{13!^4} = \frac{24\cdot 13^4\cdot 48!}{52!}$$

1.30)

(a) The probability is

$$1-\left(1-\frac{1}{6}\right)^4 = 1-\left(\frac{5}{6}\right)^4 = \cancel{\ \ } .518$$

(b) The probability is

$$1-\left(1-\frac{1}{36}\right)^{24} = 1-\left(\frac{35}{36}\right)^{24} = 0.491$$

So ~~b~~ a is greater than ~~a~~ b.

Observe that for small x we

have

$$(1-x)^n \simeq 1 - nx + O(x^2)$$

Thus at This level of appoximation a and b are equal. In othe world The pobability of <u>exactly</u> 1 six in for Throws equal the probability of <u>exactly</u> one double six in 24 double throws.

1.44) We have

$$\big(A \cap (\Omega \setminus B)\big) \cup (A \cap B) = A$$

while

$$(A \cap (\Omega \setminus B)) \cap (A \cap B) = \phi$$

So we have

$$\mathbb{P}(A \cap (\Omega \setminus B)) + \mathbb{P}(A \cap B) = \mathbb{P}(A)$$

Thus if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$ Than

we have

$$\mathbb{P}(A \cap (\Omega \setminus B)) = \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B) =$$

$$= \mathbb{P}(A)(1 - \mathbb{P}(B)) =$$

$$= \mathbb{P}(A)\mathbb{P}(\Omega \setminus B)$$

since $\mathbb{P}(\Omega \setminus B) = 1 - \mathbb{P}(B)$.

1.52(a) ~~Next~~ Define the events

$B_1 = \{$first pick is Black$\}$
$W_1 = \{$first pick is White$\}$

$B_2 = \{$second pick is Black$\}$
$W_2 = \{$second pick is White$\}$

Thus.

$$\mathbb{P}(B_2) = \mathbb{P}(B_2 \mid B_1)\mathbb{P}(B_1) +$$

$$\mathbb{P}(B_2 \mid W_2)\mathbb{P}(W_2) =$$

$$= \frac{7}{9} \cdot \frac{4}{7} + \frac{6}{9} \cdot \frac{3}{7} = \frac{46}{63}$$

(b) Define

$A_I = \{$you ~~who~~ pick urn $I\}$
$A_{II} = \{$you pick urn $II\}$

and

$$B = \{\text{you pick Black}\}$$

$$W = \{\text{you pick White}\}$$

$$P(A_I \mid B) = \frac{P(B \mid A_I)\,P(A_I)}{P(B \mid A_I)\,P(A_I) + P(B \mid A_{II})\,P(A_{II})}$$

$$= \frac{\frac{4}{7} \cdot \frac{1}{2}}{\frac{4}{7} \cdot \frac{1}{2} + \frac{6}{8} \cdot \frac{1}{2}} = \frac{16}{37}$$

Problems

9. Comments on The solution in The book:

a) ~~Th~~ If The coin is fair The prob. of getting K head is equal To The prob of getting K Tail, That is n-K head. Thus

$$\mathbb{P}(Y=k) = \mathbb{P}(Y=n-k)$$

b) If X and Y are 2 r.v we have

$$\mathbb{P}(X+Y=k) = \sum_{\substack{x,y \\ x+y=k}} \mathbb{P}(X=x)\mathbb{P}(Y=y) =$$

$$= \sum_{x} \mathbb{P}(X=x)\mathbb{P}(Y=k-x)$$

14) a) Assume The formula is True for $n \leq N$ we will prove it for $n = N$.

We can rewrite The equation as

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{I \subset \{1, \dots n\}} (-1)^{|I|-1} \mathbb{P}\left(\bigcap_{i\in I} A_i\right)$$

where The sum is over all subseT of $\{1, 2, \dots, n\}$ and $|I|$ is The cardinality of I (i.e. The number of elemenTs in I).

$$\mathbb{P}\left(\bigcup_{i=1}^{N} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{N-1} A_i\right) + \mathbb{P}(A_N) - \mathbb{P}\left(\left(\bigcup_{i=1}^{N-1} A_i\right) \cap A_N\right)$$

but

$$\left(\bigcup_{i=1}^{N-1} A_i\right) \cap A_N = \bigcup_{i=1}^{N-1} (A_i \cap A_N)$$

so that using the inductive assumption we get

$$\mathbb{P}\left(\bigcup_{i=1}^{N} A_i\right) = \mathbb{P}(A_N) + \sum_{I \subset \{1, \dots N-1\}} (-1)^{|I|-1} \mathbb{P}\left(\bigcap_{i \in I} A_i\right)$$

$$- \sum_{I \subset \{1, \dots N-1\}} (-1)^{|I|-1} \mathbb{P}\left(\bigcap_{i \in I} (A_i \cap A_N)\right)$$

but

$$\bigcap_{i \in I} (A_i \cap A_N) = \left(\bigcap_{i \in I} A_i\right) \cap A_N$$

Thus the last sum becomes

$$- \sum_{I \in \{1, \dots N-1\}} (-1)^{|I|-1} \mathbb{P}\left(\bigcap_{i \in I} (A_i \cap A_N)\right) = \sum_{\substack{J \subset \{1 \dots N\} \\ N \in J}} (-1)^{|J|-1} \mathbb{P}\left(\bigcap_{j \in J} A_j\right)$$

The statement easily follows.

b) Let

$$A_i = \{ i\text{th Key is on its hook} \}$$

we need

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right)$$

but $\mathbb{P}\left(\bigcap_{i \in I} A_i\right) = \left(\frac{1}{n}\right)^{|I|}$ so that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{n^k} =$$

$$1 - \left(1 - \frac{1}{n}\right)^n = 1 - \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right)$$

Clearly

$$\lim_{n \to \infty} \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = 1 - e^{-1}$$

The following morning we have the same situation but this time

$$\mathbb{P}\left(\bigcap_{i=I} A_i\right) = \frac{(n-|I|)!}{i!}$$

since no two keys can be on the same hook. Thus

$$\mathbb{P}\left(\bigcup_i A_i\right) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{(n-k)!}{n!} =$$

$$= 1 - \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$$

So that again

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcup_i A_i\right) = 1 - e^{-1}.$$

Since the probability that no key was hung on its own hook is

$$\mathbb{P}\left(\bigcap_i A_i^c\right) = 1 - \mathbb{P}\left(\bigcup_i A_i\right) = \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$$

so that

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcap_i A_i^c\right) = e^{-1}$$

17) If the first Toss is head there are 2 possibilities:

1) iT is followed by $r-1$ heads

2) There is at least one Tail in the following $r-1$ Tosses

In The first case we have a success. In The second case we can strant again after we see the first Tail. Observe that, given That There is at least one Tail in The first $r-1$ Tosses, The Tosses after The first Tail are independent. Thus we get

$$P(E \mid A = \text{head}) = p^{r-1} + (1-p^{r-1})\, P(E \mid A = \text{Tail})$$

On the other hand if the first toss in tail, there need to be at least ~~another~~ a head in the following s-1 tosses so that

$$P(E \mid A = tail) = (1 - q^{s-1})\, P(E \mid A = head)$$

Thus

$$P(E) = P(E \mid A = tail)\, q + P(E \mid A = head)\, p$$

but

$$P(E \mid A = head) = p^{r-1} + (1 - p^{r-1})(1 - q^{s-1})\, P(E \mid A = head)$$

so that

$$P(E \mid A = head) = \frac{p^{r-1}}{1 - (1 - p^{r-1})(1 - q^{s-1})}$$

$$P(E \mid A = tail) = \frac{(1 - q^{s-1})\, p^{r-1}}{1 - (1 - p^{r-1})(1 - q^{s-1})}$$

$$P(E) = \left(p^{r} + q(1 - q^{s-1})\, p^{s-1}\right) \Big/ \left(p^{r-1} + q^{s-1} - p^{r-1} q^{s-1}\right)$$