Journal of Economic Theory 106, 417–435 (2002)
doi:10.1006/jeth.2001.2897

Statistical Equilibrium Wealth Distributions in an Exchange Economy with Stochastic Preferences

Jonathan Silver[1]

Laboratory of Molecular Microbiology, National Institute of Allergy and Infectious Disease, National Institutes of Health, Bethesda, Maryland 20892
jsilver@nih.gov

Eric Slud

Department of Mathematics, University of Maryland, College Park, Maryland 20742

and

Keiji Takamoto

Laboratory of Molecular Microbiology, National Institute of Allergy and Infectious Disease, National Institutes of Health, Bethesda, Maryland 20892

Received January 2, 2001; final version received June 25, 2001

We describe an exchange market consisting of many agents with stochastic preferences for two goods. When individuals are indifferent between goods, statistical mechanics predicts that goods and wealth will have steady-state gamma distributions. Simulation studies show that gamma distributions arise for a broader class of preference distributions. We demonstrate this mathematically in the limit of large numbers of individual agents. These studies illustrate the potential power of a statistical mechanical approach to stochastic models in economics and suggest that gamma distributions will describe steady-state wealths for a class of stochastic models with periodic redistribution of conserved quantities. *Journal of Economic Literature* Classification Numbers: C15, C62, C73, D3, D5.

Key Words: equilibrium; wealth distribution; stochastic preferences; gamma distribution; Markov model; exchange market.

1. INTRODUCTION

Academic models of financial markets since the beginning of the 20th century have noted the essentially random nature of successive changes in

[1] To whom correspondence should be addressed. Fax: (301)402-0226.

0022-0531/02 $35.00

asset prices. Yet classical economic theory describes deterministic transitions of the detailed states of an economic system which balance supply and demand in response to changes in preference and in information (e.g., in expectations of future returns). This dichotomy between (largely) deterministic detailed transitions and stochastic observed aggregates in economics [1, 2] is analogous to that in physics between motions of Newtonian particles and macro- level observations of Brownian motion and thermodynamic variables [3–5]. In both economics and physics, the limiting behavior for superposed effects of large numbers of individuals (particles) is due to probabilistic central limit theorems. The first such theoretical results asserted Gaussian-distributed displacements [1–4]. More realistic models in physics, going beyond Gaussian observations, have incorporated interactions among particles [5]. In economics, statistical description of aggregates based on probabilistic limit theorems has been broadened to include infinite-variance laws and stable-increments processes [6–8]. Economic models leading to non-Gaussian distributions often involve complex or heterogeneous interactions among market participants [9–11], but results with true statistical-mechanics flavor, deriving such behavior from detailed microeconomic-level mechanisms, are rare.

One early attempt to incorporate thermodynamic and statistical mechanical reasoning in economics [12] was largely confined to analogy without detailed mathematical content. Another approach [13], which we largely follow, formulates the problem of economic market equilibrium as that of finding a time-invariant, nondegenerate probability distribution in a high-dimensional state space describing the fortunes of a large population of firms and individuals. The authors of [13] propose that microeconomic production decisions be treated deterministically while market fluctuations based on changing preferences and exogenous influences be considered a stochastic process. Reference [2] provides one of the few cases of analysis with simplifying large-population approximations leading to true invariant distributions of a large state-space Markov model in economics. There are several references, and a growing physics literature, with statistical mechanics flavor in economic modeling, which describe equilibria via maximum "entropy" configurations subject to constraints such as fixed total wealth [14–16].

In this paper, we adopt a combined theoretical and simulation approach to a simple model in which a fixed supply of goods is traded among individuals whose preferences for goods are determined stochastically. A central feature of our model is that individuals periodically change their preferences for goods independently, according to a distribution of preferences in the population. The changes in aggregate demand translate in the next time-period into changes in the market-clearing price. As individuals adjust the mix of assets they hold to satisfy their changing preferences, the value of their goods waxes and wanes according to how well, by chance, they anticipate future demand.

Using four separate lines of argument, of increasing generality, we show that the model results in a probabilistic steady state, independent of initial conditions, with gamma distributions of goods and wealths. The first is a simulation study, which confirmed the steady-state behavior of the model under various assumed preference distributions and showed that wealths and amounts of goods behaved like gamma random variables with parameters depending only on the preference distribution. The second line of argument follows the heuristic of [14] from statistical physics in calculating maximum "entropy" configurations subject to the constraint of fixed total wealth. This argument is applicable only when individuals are indifferent between goods, in which case we, like [14], treat all microstates as equiprobable because all microstates accessible from each wealth-configuration are. The third line of argument is a simple consequence of a theorem of [17]: gamma-distributed wealths are implied by the assumption that in steady state, wealths are jointly independent with ratios independent of their sums. Our final argument, a mathematical development along the lines of the approach to statistical mechanics in [4], shows how the gamma distribution for wealths follows necessarily from the assumption that the individual wealths have steady-state distributions characteristic of a set of independent identically distributed variables conditionally given their sum. The generality of our result that wealths have steady-state gamma distributions, across our family of models, suggests that it may be applicable to a variety of stochastic models in which conserved quantities are periodically re-balanced.

2. THE MODEL

For simplicity, we begin by considering a market consisting of N individuals and two goods, A and B, the total number of units of each being conserved respectively at αN and βN. At the start, we assign to individuals (indexed by i) certain amounts a_{i0} of good A and b_{i0} of good B. At all integer times t, individuals hold amounts of good A and B respectively denoted a_{it} and b_{it}, and measure their total wealths in terms of an external numéraire, such as gold or inflation-adjusted specie, the total quantity of which is conserved, at ωN. We use such a"gold standard" in order to maintain symmetry in the treatment of goods A and B, and with a view to allowing many more goods in further developments of the model. A somewhat simpler and less artificial approach, which we also follow in defining notations, is to view units of A as numéraire, with ϑ_t denoting the time-dependent value of one unit of good B in units of good A at time t.

Initially, the holdings a_{i0}, b_{i0} are assigned; thereafter, at each time-period $t \geqslant 1$, individuals determine fractions of their wealths

$$w_{it} = a_{i,t-1} + \vartheta_t b_{i,t-1} \tag{1}$$

to allocate to good A, with the remainder allocated to good B. This is equivalent to maximizing over $a_{it} \in [0, w_{i,t-1}]$ an individual Cobb–Douglas [18] utility function of the form

$$U_{it}(a_{it}) = (a_{it})^{f_{it}} (w_{it} - a_{it})^{1-f_{it}}, \tag{2}$$

where f_{it} is a doubly indexed sequence of independent and identically distributed random variables with values in the interval [0, 1]. The maximization of (2) results in the assignment

$$\left.\begin{aligned} f_{it} w_{it} &= a_{it} \\ (1-f_{it})\, w_{it} &= \vartheta_t b_{it}. \end{aligned}\right\} \tag{3}$$

The so-far-undetermined price ϑ_t is uniquely determined from variables at time $t-1$ together with $\{f_{it}\}$ via the market-clearing constraint

$$\sum_{i=1}^{N} a_{it} = \alpha N, \qquad \sum_{i=1}^{N} b_{it} = \beta N. \tag{4}$$

This constraint immediately yields

$$\sum_{i=1}^{N} f_{it} w_{it} = \alpha N, \qquad \sum_{i=1}^{N} (1-f_{it})\, w_{it} = \vartheta_t \beta N \tag{5}$$

from which there follows, upon substituting (1) for w_{it} and solving for ϑ_t,

$$\vartheta_t = \sum_{i=1}^{N} (1-f_{it})\, a_{i,t-1} \Big/ \sum_{i-1}^{N} f_{it} b_{i,t-1}. \tag{6}$$

Substituting this expression for ϑ_t back into (3) leads to the expression

$$\left.\begin{aligned} a_{it} &= f_{it} a_{i,t-1} + f_{it} b_{i,t-1} \textstyle\sum_j (1-f_{jt})\, a_{j,t-1} / \sum_j f_{jt} b_{j,t-1} \\ b_{it} &= (1-f_{it})[b_{i,t-1} + a_{i,t-1} \textstyle\sum_j f_{jt} b_{j,t-1} / \sum_j (1-f_{jt})\, a_{j,t-1}]. \end{aligned}\right\} \tag{7}$$

Thus, individuals' holdings of goods (7), wealth (1), and prices (6) at time t can be expressed in terms of their previous holdings $(a_{i,t-1}, b_{i,t-1})$ and the N random variables $\{f_{it}\}$, making this a Markov process discrete in time with $2N$ continuous state-variables a_{it} and b_{it} (which satisfy two linear constraints, making $2N-2$ the effective state dimension).

The equations determining new state from old have the alternative expression (6) together with

$$\begin{pmatrix} a_{it} \\ b_{it} \end{pmatrix} = \begin{pmatrix} f_{it} \\ (1-f_{it})/\vartheta_t \end{pmatrix} \begin{pmatrix} 1 \\ \vartheta_t \end{pmatrix}' \begin{pmatrix} a_{i,t-1} \\ b_{i,t-1} \end{pmatrix} \tag{8}$$

or, after premultiplying the last equation by the row-vector $(1, \vartheta_{t+1})$,

$$w_{i,t+1} = \left(f_{it} + \frac{\vartheta_{t+1}}{\vartheta_t}(1-f_{it}) \right) w_{it}. \tag{9}$$

If values are expressed in units of a "money" numéraire of which there is a total quantity ωN, the aggregate relations (5) imply that the total of $(\alpha+\beta\vartheta_t)\,N$ units of numéraire A at time t has value ωN, so that 1 unit of A has monetary value $\omega/(\alpha+\beta\vartheta_t)$ at time t, and all prices in units of A are converted to money prices through multiplication by the factor $\omega/(\alpha+\beta\vartheta_t)$. The wealth w_{it} of agent i at time t in terms of units of A is then equivalent to the "money wealth:"

$$m_{it} \equiv w_{it}\omega/(\alpha+\beta\vartheta_t).$$

The simulation results in the next section are presented in terms of money-wealths. As will be seen in Section 6 below, when N is large the ratio $\omega/(\alpha+\beta\vartheta_t)$ is distributionally indistinguishable from the constant 1, and thereore m_{it} and w_{it} have asymptotically the same distribution.

3. SIMULATION RESULTS

We began by examining, in computer simulations with two assets (A and B) and $N = 1000$ individuals, whether our model with uniform preference distribution leads to a steady state and a recognizable limiting distribution for individual wealths. In our simulations, the assets were initially either equally divided ($a_{i0} = b_{i0} = 1/N$ for all i) or alternatively 99.9 % of A and B were initially in the hands of one individual with the remaining assets equally divided. We characterized the distribution of goods and wealth every 50 cycles by evaluating their first five moments, i.e. the average over individuals of a_i^k, b_i^k, and money-wealths m_i^k, for $k = 1$ to 5. After about $10 \cdot N$ (i.e., 10,000) cycles, the moments appeared to have reached steady-state values that were independent of the initial conditions. This steady state is summarized in Table I, which gives average moments and standard deviations, where averages were taken over 200 cycles (every 50th cycle from cycle 15,050 to 25,000). Since the moments were essentially identical for

TABLE I

Measured and Predicted Values of Weighted kth Moments and v_k for Steady-State Goods and Money Wealth Distributions Arising from Uniform Preferences

k	Weighted kth moment			v_k computed from moments		
	Theory	Simulation		Theory	Simulation	
		Mean	St. dev.		Mean	St. dev.
			Goods			
1st	1	1.00				
2nd	2	1.97	0.06			
3rd	6	5.79	0.54	1	1.06	0.14
4rd	24	26.6	4.9	1	1.14	0.27
5th	120	111	48	1	1.31	0.41
			Wealths			
1st	1	1.00				
2nd	1.5	1.48	0.03			
3rd	3	2.90	0.17	2	2.09	0.24
4rd	7.5	7.14	0.9	2	2.14	0.41
5th	22.5	21.1	5.0	2	2.30	0.63

Note. Theoretical values in columns 2 and 5 are $v^{-k}\Gamma(v+k)/\Gamma(v)$ and v for the gamma distribution ($v=\gamma=1$) for goods and ($v=\gamma=2$) for wealths. Simulation results are mean and standard deviation for weighted moments in columns 3 and 4, and v_k (see text for definition) in columns 6 and 7, calculated over every 50th cycle of the data from 15,050 to 25,000.

the a_i and b_i distributions, as expected by symmetry, only one set of moments is listed (goods distribution). The moments scaled as α^{-k} for goods and ω^{-k} for wealths; to eliminate these scaling factors, Table I gives weighted moments, equivalent to choosing $\alpha=\omega=1$. The Gamma(v, γ) probability density, with shape-parameter v, scale parameter γ, and mean v/γ, is defined as $p(x)=(\gamma^v/\Gamma(v))\,x^{v-1}\exp(-x\gamma)$ for $x>0$, and we found that the moments of the empirical goods distribution were in excellent agreement with those of Gamma($1, 1/\alpha$), while the empirical moments of (monetary) wealths agreed closely with those of Gamma($2, 2/\omega$) (Table I, column 3).

We also performed simulations using nonuniform preference distributions. In this case, the simulation results suggested that the steady-state goods and wealth distributions were still closely approximated by gamma distributions, although the parameter v was no longer an integer. We performed simulations with two types of preference distributions: truncated Gaussians, and polynomial densities of the form $p(f) \propto (\min(f, 1-f))^c$

for $0 < f < 1$, where $c = 1, 2, 3, 4, 5, -1/2, -2/3, -3/4$, and $-4/5$. The Gamma$(v, v/\omega)$ distribution with mean ω and shape parameter v has as kth moment $\langle m^k \rangle = (v/\omega)^{-k}\,\Gamma(v+k)/\Gamma(v)$. It follows that the weighted ratio of moments $(1/\omega)\langle m^k \rangle / \langle m^{k-1} \rangle$ increases by $1/v$ when k increases by 1. In the simulations, the values of v_k calculated from the weighted ratios of moments were fairly constant for successive values of k (Table I, column 6, and Table II), which suggests that the distributions are close to Gamma.

TABLE II

Estimates of v_k, Calculated from Differences in Successive Moment Ratios

Preference dist'n	Goods dist'n		Wealth dist'n	
	Mean	St. dev.	Mean	St. dev.
	Gaussian, $\sigma = 0.125$			
v_1	8.03	0.69	17.4	2.34
v_2	8.13	0.85	16.7	1.77
v_3	8.05	1.08	16.8	1.79
	$p(f) = 12f^2, 0 \leqslant f \leqslant 0.5$			
v_1	4.56	0.41	9.03	0.76
v_2	4.49	0.66	8.90	1.00
v_3	4.45	0.95	8.53	1.13
	$p(f) = 4f, 0 \leqslant f \leqslant 0.5$			
v_1	2.50	0.26	4.82	0.32
v_2	2.51	0.46	4.67	0.64
v_3	2.63	0.68	4.60	1.06
	Gaussian, $\sigma = 0.25$			
v_1	2.14	0.23	4.30	0.29
v_2	2.17	0.38	4.41	0.47
v_3	2.31	0.57	4.62	0.67
	Gaussian, $\sigma = 0.5$			
v_1	1.28	0.16	2.50	0.19
v_2	1.36	0.27	2.56	0.35
v_3	1.56	0.42	2.72	0.59
	$p(f) = 1/\sqrt{8f}, 0 < f \leqslant 0.5$			
v_1	0.45	0.09	0.88	0.12
v_2	0.53	0.16	0.98	0.22
v_3	0.69	0.25	1.17	0.35
	$p(f) = (108f^2)^{-1/3}, 0 < f \leqslant 0.5$			
v_1	0.31	0.07	0.58	0.10
v_2	0.38	0.13	0.67	0.19
v_3	0.52	0.19	0.86	0.31

Note. "Gaussian" with indicated standard deviation σ is truncated, i.e., conditioned to lie in [0, 1]. The power-law densities were all taken symmetric about 0.5.

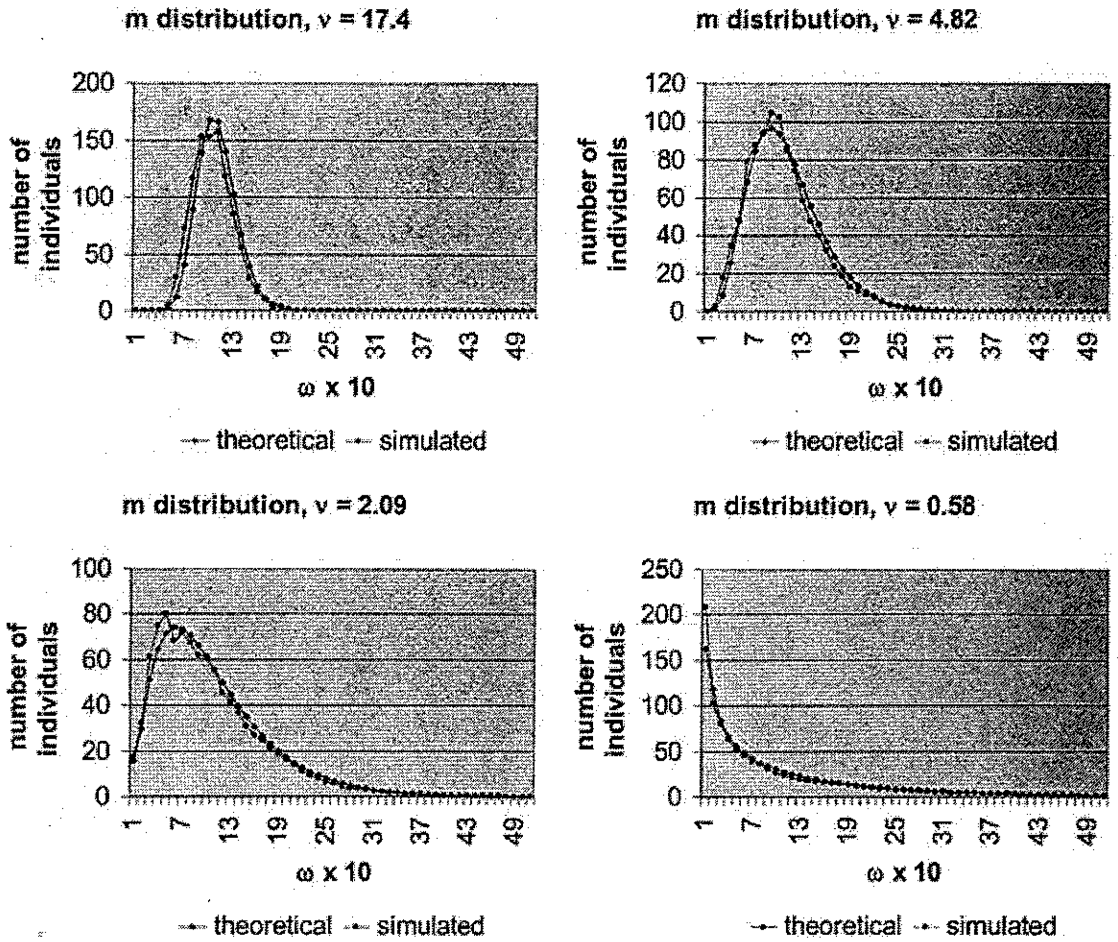

FIG. 1. Comparison of theoretical and simulated wealth distributions for some of the preference distributions shown in Tables I and II. Values of ν were taken from the simulation results as in Table II.

Figure 1 shows graphically, in the form of slightly smoothed histograms plotted in *Excel* overlaid with gamma densities with mean ω and shape factors ν taken from Tables I and II, that the steady-state wealth distributions from the simulation data were very close to gamma. While the simulation results are suggestive, they do not prove that the wealths asymptotically follow gamma distributions. Hence, we sought theoretical arguments.

4. HEURISTIC ARGUMENT COUNTING MICROSTATES

The nonlinearity of Eq. (7) makes the exact analysis of this system intractable. However, for the particular case of a uniform preference distribution for f_{it} for all i and t, with density $h(f) = 1$ for $0 < f < 1$, the asymptotic behavior of the system for large N and t can be roughly understood by arguing as in [14]. In the following heuristic argument, there is a close analogy between the distribution of wealth in the economic system and the distribution of energy among molecules of an ideal gas in statistical mechanics. In the latter case all "microstates" (specifications of the energy of each molecule) with the same total energy are deemed equally likely. One considers the number of ways Ω in which one can have N_j molecules with energy E_j, subject to the constraints that the total energy and number of molecules are fixed. The constraints are handled via Lagrange multipliers and the distribution of energy is determined by maximizing the function

$$G(N_j, E_j) = \log \Omega - \lambda_1 \left(\sum_j N_j - N \right) - \lambda_2 \left(\sum_j N_j E_j - E \right)$$

with respect to the N_j (see [19]). In the case of the economics model with uniform preferences, all ways of dividing an individual's wealth between goods A and B are equally likely. If wealth is measured in discrete units (e.g., dollars), the number of ways of allocating money-wealth m_i between two goods is proportional to $m_i + 1$ (zero dollars to good A, or one, or two, ... up to m_i dollars). Hence, the number of ways one can have N_1 individuals with wealth m_1, N_2 individuals with wealth m_2, etc., is $\Omega(N_j, m_j) = N! \Pi_j((m_j + 1)^{N_j}/N_j!)$. The factorials account for the fact that different combinations of individuals in a set of wealth bins represent different microstates. There are $N!$ permutations of individuals, but this overcounts the number of microstates because permutations of individuals within a wealth bin do not represent different microstates; the $N_j!$ terms in the denominator correct for this overcounting. The N_j can be made independent variables by the introduction of Lagrange multipliers λ_1 and λ_2 corresponding to the constraints that the number of individuals and the total wealth are constant. Maximizing the function $G(N_j, m_j) = \log \Omega(N_j, m_j) - \lambda_1(\sum_j N_j - N) - \lambda_2(\sum_j N_j m_j - N\omega)$ with respect to the N_j, and then solving for λ_1 and λ_2 from the constraint equations, leads, in the limit as $N_j \to \infty$ and sums are replaced by integrals, to the Gamma$(2, 2/\omega)$ density $p(m) = (2/\omega)^2 m \cdot \exp(-2m/\omega)$ for wealths. For the case of ν goods instead of two goods, this generalizes to Gamma$(\nu, \nu/\omega)$. The derivation of the distribution of goods (a_{it} or b_{it}) among individuals follows the same argument except that the term analogous to $(m_j + 1)^{N_j}$ is omitted because there is no degeneracy in the way the goods can be allocated; this leads to $\Omega(N_j, m_j) = N!/\Pi_j N_j!$ and to a Gamma$(1, 1/\alpha)$ density $p(m)$.

5. A QUALITATIVE ARGUMENT

In this section, we apply a purely probabilistic result to our model, under an appealing qualitative assumption, to derive the gamma distribution for steady-state wealths m_{it}. The assumption is that arbitrary subsets of finitely many wealth-variables at a time behave as though a steady-state distribution exists (for large t), and that

(a) the wealth-variables in the subset are asymptotically independent (for large numbers N of agents and large time-index t), i.e., the total-wealth asymptotically for large N imposes no constraint upon them, and

(b) the random mechanism of wealth-partition asymptotically makes the total wealth in a sub-population of individuals $\{i_1, i_2, ..., i_k\}$ approximately independent of the way in which that wealth is subdivided among the individuals in that sub-population.

Then the conclusion is that no distribution for the wealth-variables except gamma is possible.

In many economic contexts, agents form coalitions or groupings. Here, groups of finitely many agents are artificial, since the underlying model contains no mechanism for cooperation in the setting of preferences. (However, such a mechanism would be interesting to include in later refinements of the model.) The assumption (a)–(b) above says that the aggregate wealth for a finite grouping of agents fluctuates statistically (in steady state) in a way which is statistically unrelated to the fluctuating partition of the grouping's wealth among its individual members.

The underlying probabilistic result is the following theorem of [17]:

THEOREM 1. *If X, Y are independent positive random variables such that $X/(X+Y)$ and $X+Y$ are independent, then X, Y are respectively distributed as Gamma(v_1, γ), Gamma(v_2, γ) for positive constants v_1, v_2, γ.*

This theorem of Lukacs [17] was cited for economic relevance in a rather different context by Farjoun and Machover [13, pp. 63–72]. They were interested in the random variables profit-rate R and labor cost rate Z per unit of capital per unit time in an economy, obtained by aggregating over all firms. They adduced Theorem 1 to conclude that these variables are approximately Gamma distributed because they found it qualitatively and empirically plausible (after some discussion) that the ratio R/Z of the gross "return to capital" R over "return to labor" Z should fluctuate approximately independently of $R+Z$ (the total return excluding rent). Farjoun and Machover recognized that to apply Lukacs's Theorem in this setting, R and Z would have to be approximately independent, but they did not attempt to justify this independence.

The formal result which we derive from Theorem 1 is as follows.

THEOREM 2. *Suppose that the array of wealth variables w_{it} (implicitly also indexed by the number of agents N in our model) is such that for large*

N, t the joint distribution of each finite nonrandom set $\{i_1, ..., i_k\}$ of the random variables $(w_{i_1,t}, w_{i_2,t}, ..., w_{i_k,t})$ converges (in the usual sense of pointwise convergence of the joint distribution functions to a proper joint distribution function at continuity points of the latter), and that under the limiting distribution of $(W_1, ..., W_k)$, the variables W_j are positive and jointly independent, and

$$S_k \equiv W_1 + \cdots + W_k \qquad \text{is independent of } \left(\frac{W_1}{S_k}, \frac{W_2}{S_k}, ..., \frac{W_{k-1}}{S_k}\right).$$

Then all of the limiting W_j variables are gamma distributed with the same scale parameter $\gamma > 0$.

Proof. For each fixed $j = 1, ..., k$, $X = W_j$ and $Y = \sum_{i=1}^k W_i - W_j$ are independent random variables such that $X/(X+Y)$ is independent of $X+Y$. Thus Theorem 1 implies that W_j and $\sum_{i=1}^k W_i - W_j$ are gamma distributed with the same scale parameter $\gamma > 0$. Since it is well known that the sum of independent gamma variables with common scale parameter is gamma with the same scale, $\sum_{i=1}^k W_i$ is gamma with scale parameter γ, which is therefore the same for all j, and our proof is complete. ∎

The limitation of this approach is that the appealing assumption of independence of sums and ratios of wealths of subsets is not readily deducible from our model.

6. PROBABILISTIC THEORY

For discrete-time Markov processes with continuous state-spaces, such as the $2N-2$ dimensional process defined in (4) and (7) above, there is a fully developed but not very well-known theory of long-run asymptotics in terms of equilibrium or stationary distributions, for which see the book [20]. Under various sets of conditions, which are not easy to apply in the present context of the process defined by Eq. (7), such continuous-state processes can be shown to possess a unique stationary distribution which can be found by solving a high-dimensional integral equation. The challenge for application of this structure to economic theory is to obtain simple approximate conclusions for observables such as the distributions or moments of the wealth-variables w_{it} or m_{it}.

The mathematical argument of this section proceeds in three main steps. Because of space limitations, we describe the main steps but omit some mathematical details. We assume throughout that for large N and t

$$\frac{1}{N} \sum_{i=1}^{N} w_{it}^2 (1 - f_{it})^k \to \mu_{2,k}, \qquad k = 0, 1, 2 \tag{A.1}$$

with probability 1, where the constant limits $\mu_{2,k}$ do not depend upon N or t (but definitely do depend on k and the distribution of f_{it}). Denote

$$\mu_2 \equiv \mu_{2,0} = \lim_{N,t\to\infty} N^{-1} \sum_{i=1}^{N} w_{it}^2, \qquad \sigma^2 = \mathrm{Var}(f_{it}).$$

A further technical assumption required throughout is that the density $h(f) \equiv p_f(f)$ has finite third moment and has Fourier transform absolutely integrable on the real line.

First we apply a refinement of Central Limit Theory (the Edgeworth expansion [21], pp. 533*ff*, as generalized for non-*iid* summands in [22]) to establish the form of the conditional density of normalized price deviations $X_{t+1} \equiv \sqrt{N}\,(\vartheta_{t+1} - \vartheta)$ up to terms of order $N^{-1/2}$, where

$$\vartheta = \alpha(1-\mu)/(\beta\mu), \qquad \mu = E(f_{it}).$$

Second, we apply related limit theory along with an auxiliary assumption (A.2 below) to provide, again up to terms of order $N^{-1/2}$, the joint density of (X_t, w_{1t}, f_{1t}). Third, we show that gamma is the only possible form for asymptotic marginal distributions for w_{1t} which is time-stationary within the model (7), that is, for which the marginal distribution of $w_{1,t+1}$ within $(X_{t+1}, w_{1,t+1}, f_{1,t+1})$ is the same up to order $1/\sqrt{N}$ as that of w_{it}.

For the precise formulation of the first step, consider N large, and calculate from equations (6), (5), and (3), that

$$\begin{aligned}\frac{\vartheta_{t+1}}{\vartheta_t} &= \frac{\sum_j (1-\mu(f_{j,t+1}-\mu))\, f_{jt} w_{jt}}{\sum_j (f_{j,t+1}-\mu+\mu)(1-f_{jt})\, w_{jt}} \\ &= \frac{\vartheta}{\vartheta_t}\left\{1 - \frac{1}{\alpha N(1-\mu)} \sum_j (f_{j,t+1}-\mu)\, f_{jt} w_{jt}\right\} \\ &\quad\times\left\{1 + \frac{1}{\beta N \mu \vartheta_t} \sum_j (f_{j,t+1}-\mu)(1-f_{jt})\, w_{jt}\right\}^{-1}.\end{aligned}$$

Now, applying Theorem 19.3 of [22] to the nonidentically distributed but conditionally independent summands, given $\{f_{jt}, w_{jt}\}_{j=1}^{N}$, yields the conclusion that the conditional density (at y) of X_{t+1} given $(X_t, \{w_{jt}, f_{jt}\})$ (at $X_t = x$) is, asymptotically for large N,

$$\frac{\beta\mu}{\sigma\sqrt{\mu_2}}\,\phi\left(\frac{y\mu\beta}{\sigma\sqrt{\mu_2}}\right)\left(1 + \frac{c_3(y)}{\sqrt{N}}\right)\exp\left(\frac{\beta x}{\sqrt{N}}\, q_2(y)\right) + o(N^{-1/2}), \qquad (10)$$

where ϕ is the standard normal density, c_3 is a cubic polynomial, q_2 is a quadratic, and $o(N^{-1/2})$ denotes terms of order less than $N^{-1/2}$. In particular

$$X_t \equiv \sqrt{N}\,(\vartheta_t - \vartheta) \overset{\mathscr{D}}{\approx} \mathcal{N}\left(0, \frac{\sigma^2 \mu_2}{\beta^2 \mu^2}\right). \tag{11}$$

Thus, the central limit theorem implies that for large N, prices will be normally distributed, and further shows how the variance of prices depends on the mean and variance of preferences and the variance of wealths. In conformity with the assertion (11) of asymptotic normality, we remark that when histograms were made from ϑ_t variables simulated (with $N \geqslant 100$, and $f_{it} \sim$ *Uniform*$[0, 1]$) from the model under study, these histograms looked approximately normal, but the histograms of the variables $\log \vartheta_t$ looked still more symmetric and therefore closer to normality. Furthermore, as expected, in the limit as $N \to \infty$, the prices become fixed. Since price fluctuations drive the redistribution of wealth, it may seem puzzling that the initial wealth distribution is significantly altered for very large N. The explanation is that the number of time-steps required to reach the new steady state increases with N, and is of the order of N according to the simulation results.

The next step relies on an auxiliary assumption restricting the form of the conditional density of (w_{1t}, f_{1t}) given X_t (or equivalently, given ϑ_t), namely:

(A.2) The variables w_{it} have a "nonlattice distribution [22, condition (19.29)] with third moments bounded uniformly in N, t; and asymptotically for large N, t, conditionally given ϑ_t, the 2-vectors $(w_{it}, w_{it} f_{it})'$ behave as N independent identically distributed random vectors subject to

$$N^{-1} \sum_{i=1}^{N} \begin{pmatrix} w_{it} \\ w_{it} f_{it} \end{pmatrix} = \begin{pmatrix} \alpha + \beta \vartheta_t \\ \alpha \end{pmatrix}. \tag{12}$$

This assumption can be compared with, but is quite different in spirit from, the independence hypothesis of Theorem 2. In our model, the conditions (12) impose a specific dependence on the wealths w_{it}. However, as the number N of agents gets large, nonrandom finite subsets of them may have wealths which are hardly affected by the constraints (12) and may therefore be approximately independent, as assumed in Theorem 2. The alternative approach of **(A.2)** is to argue that the variables w_{it} may still behave independently *conditionally* given the sum-constraints. This latter approach is motivated as in classical statistical mechanics (cf. [4]), where the exact equilibrium distribution of particle positions and momenta is intractable but the individual phase-space coordinates behave like independent random variables conditionally given a sum-condition fixing energy and possibly some macroscopic integrals of the motion of a dynamical system.

By use of the same non-*iid* Edgeworth-expansion result of [22] cited above, Assumptions **(A.2)** plus **(A.1)** lead to the following technical conclusion: For a family of positive differentiable functions g_N on $[0, \infty)$ such that the integrals $\int w^3 g_N(w)\, dw$ are uniformly bounded, and for some functions c_1, c_2, c_3 which may depend upon x,

$$p_{w_{1t}, f_{1t} \mid X_t}(w, f \mid x) = h(f)\, g_N(w) \left(1 + \frac{c_1 w + c_2 w(f-\mu) + c_3}{\sqrt{N}} \right) + o(N^{-1/2}), \tag{13}$$

where the final o term is understood as a function $r_N(f, w, x)$ such that for each x, as N gets large, $N^{1/2} \int_0^1 \int_0^\infty w^3 \,|r_N(f, w, x)|\, dw\, df \to 0$.

In fact, the derivation of (13) from **(A.2)** shows that

$$c_1 = \frac{\beta x}{\sigma_2^2}, \qquad c_2 = \frac{-\mu\beta x}{\sigma^2 \mu_2}, \qquad c_3 = -\mu_1 c_1, \tag{14}$$

where we define for convenience

$$\mu_1 = \alpha/\mu, \qquad \sigma_2^2 \equiv \mu_2 - \mu_1^2.$$

Formula (13) turns out to be precisely what is needed for the third step of our derivation of gamma-distributed wealths. It is much less stringent than **(A.2)** in restricting only the joint behavior of (w_{1t}, f_{1t}, X_t). It can be understood as a first-order Taylor-series correction for large N to the conditional independence of w_{1t} and f_{1t} given X_t and (12). That the variables w, f appear in this correction only through terms proportional to w, wf is plausible because (12) is the model's restriction on independence of w_{1t}, f_{1t}. It can further be derived from (13) by use of **(A.1)**, (5) and symmetric permutability of the agents within the model (7), that c_1, c_2, c_3 in (13) are necessarily linear in x and have the form (14). Thus, (13) can replace **A.2** as the fundamental assumption describing the way in which wealths behave independently (to order $N^{-1/2}$) despite having a fixed sum.

Our third step is to derive and solve the stationary equation expressing the requirement that the joint density of $(w_{1,t+1}, X_{t+1})$ be the same up to terms of order $1/\sqrt{N}$ as the joint density of (w_{1t}, X). By (10), (11), and (13), letting $p_{X_t}(x)$ and $p_{X_{t+1}}(y)$ respectively denote the approximately normal densities of X_t and X_{t+1}, we find that the joint density of $(X_t, w_{1t}, f_{1t}, X_{1,t+1})$ is, up to terms of smaller order than $N^{-1/2}$,

$$p_{X_t}(x)\, g_N(w)\, h(f)\, p_{X_{t+1}}(y) \exp\left\{ \frac{\beta x}{\sqrt{N}} \left[\frac{w-\mu_1}{\sigma_2^2} - \frac{w\mu(f-\mu)}{\sigma^2 \mu_2} + q_2(y) \right] \right\}, \tag{15}$$

where $q_2(y)$ is quadratic as in (10). However, as a function of $(X_t, w_{1t}, f_{1t}, X_{t+1})$, by definition of X_t and X_{t+1} together with (9),

$$w_{1,t+1} = \left(1+\left(\frac{\vartheta+X_{t+1}/\sqrt{N}}{\vartheta+X_t/\sqrt{N}}-1\right)(1-f_{1t})\right)w_{1t}. \tag{16}$$

Our task is to compare, up to terms of order $1/\sqrt{N}$, the joint distribution of (w_{1t}, X_t) obtained from (15) with that for $(w_{1,t+1}, X_{t+1})$ obtained from (13) and (14), which at (w^*, y) is

$$g_N(w^*)\, p_{X_{t+1}}(y)\left(1+\frac{\beta y(w^*-\mu_1)}{\sigma_2^2\sqrt{N}}\right)+o(N^{-1/2}).$$

We do this by calculating $E(\psi(w_{1,t+1}, X_{t+1})$ in two ways, with respect to the two density forms just given, and equating them, where $\psi(w, y)$ is an arbitrary smooth and compactly supported function. With (16) substituted for $w_{1,t+1}$, this gives

$$\iiiint \psi\left(w\left(1-f\right)\frac{y-x}{\vartheta\sqrt{N}}, y\right)p_{X_{t+1}}(y)\, p_{X_t}(x)\, h(f)\, g_N(w)$$
$$\left\{1+\frac{\beta x}{\sqrt{N}}\left[q_2(y)+\frac{1}{\sigma_2^2}\left(w\left(1+(1-f)\frac{y-x}{\sqrt{N}}\right)-\mu_1\right)\right.\right.$$
$$\left.\left.-\frac{(f-\mu)\,\mu w}{\sigma^2\mu_2}\left(1+(1-f)\frac{y-x}{\vartheta\sqrt{N}}\right)\right]\right\}dx\,df\,dy\,dw$$
$$=\iint p_{X_{t+1}}(y)\, g_N(w)\left\{1+\frac{\beta y(w-\mu_1)}{\sigma_2^2\sqrt{N}}\right\}\psi(w, y)\, dy\, dw.$$

Now subtract from both sides of the last equation the term

$$\iint p_{X_{t+1}}(y)\, g_N(w)\, \psi(w, y)\, dy\, dw,$$

multiply through by $\sqrt{N}$, and use the approximate identity $\int x p_{X_t}(x)\, dx = O(N^{-1/2})$, to find after using smoothness of ψ to translate a difference quotient into a derivative plus remainder,

$$\iint p_{X_{t+1}}(y)\, g_N(w)\left[\frac{1-\mu}{\vartheta}(y-\mu_1)\, w\frac{\partial\psi}{\partial w}(w, y)+\psi(w, y)\frac{\beta}{\sigma_2^2}(w-\mu_1)\,\mu\right]dy\, dw$$
$$\approx\frac{\beta}{\sigma_2^2}\iint p_{X_{t+1}}(y), g_N(w)\, y(w-\mu_1)\,\psi(w, y)\, dy\, dw,$$

where $\approx$ means that the difference between the two sides converges to 0 as $N \to \infty$. This implies that

$$\iint p_{X_{t+1}}(y)\, g_N(w) \Big[(y-\mu)(w-\mu_1)\, \psi(w, y) - \frac{(1-\mu)\,\sigma_2^2}{\beta \vartheta} (y-\mu)\, w \frac{\partial \psi}{\partial w}(w, y) \Big]\, dy\, dw \approx 0.$$

Since ψ is an arbitrary smooth compactly supported function, and since integration by parts says that

$$\int w g_N(w) \frac{\partial \psi}{\partial w}(w, y)\, dw = -\int \psi(w, y)(w g_N'(w) + g_N(w))\, dw,$$

it follows that, up to terms converging to 0 as $N \to \infty$,

$$\iint p_{X_{t+1}}(y)\, g_N(w) \Big[(y-\mu)(w-\mu_1)\, \psi(w, y) + \frac{(1-\mu)\,\sigma_2^2}{\beta \vartheta} (y-\mu)\, \psi(w, y)(w g_N'(w) + g_N(w)) \Big]\, dy\, dw \approx 0.$$

Since ψ is an arbitrary smooth compactly supported function on $[0, \infty) \times [0, 1]$, we can equate 0 to the integrand of the last integrals to obtain the approximate equation

$$g_N(w) \left(\frac{\mu_1 (w-\mu_1)}{\sigma_2^2} + 1 \right) = -w g_N'(w). \tag{17}$$

All solutions to this equation have the form

$$\begin{aligned} \log g(w) &= -\int \left(\frac{\mu_1}{\sigma_2^2} + \frac{1 - \mu_1^2/\sigma_2^2}{w} \right) dw \\ &= C - \frac{\mu_1}{\sigma_2^2} w + \left(\frac{\mu_1^2}{\sigma_2^2} - 1 \right) \log w \end{aligned}$$

for some constant C, and the only such solution g which is a probability density on $[0, \infty)$ is the Gamma$(\mu_1^2/\sigma_2^2, \mu_1/\sigma_2^2)$ density, which has mean μ_1 and variance $\sigma_2^2 = \mu_2 - \mu_1^2$.

Thus, the equality up to order $N^{-1/2}$ terms of (expectations with respect to) the distributions of (w_{1t}, X_t) and of $(w_{1,t+1}, X_{t+1})$ has been shown to imply that the wealth variables $w_{1,t+1}$ must be approximately gamma. The second moment of w_{1t} is not predicted from this development, although it is connected to the limiting variance of the variables $\sqrt{N}(\vartheta_{t+1}-\vartheta)$ through

$$\text{Asymptotic variance of } \sqrt{N}\,(\vartheta_{t+1}-\vartheta)=\frac{\sigma^2\mu_2}{\beta^2\mu^2}.$$

7. DISCUSSION

We simulated the evolution of artificial markets in which individuals buy and sell goods in order to achieve a certain fraction f of their wealth in each type of good. In our model, the prices of goods were determined endogenously to equate fixed supply with variable demand. In contrast to an analysis in which individual preferences are fixed, we randomized preferences periodically, with preference probabilities determined by various distribution functions. The purpose of the repeated randomization was to see how stochastic forces would alter the distribution of goods and wealth. While goods and wealth were continuously exchanged among individuals, their distributions came to an equilibrium, independent of starting conditions, determined by the form of the distribution of preferences for good A versus B.

Remarkably, the equilibrium densities of goods and wealth had the same mathematical form $p(x)=\gamma^\nu x^{\nu-1}e^{-\gamma x}/\Gamma(\nu)$ for all preference distributions tested. While this form was suggested for the case of uniform preferences, it was not obvious that it should hold for non-uniform preferences. Nevertheless, we showed by an extension of the central limit theorem that if individual wealths behave in steady state as identically distributed, independent variables conditioned on their fixed sum, then all preference functions satisfying reasonable continuity conditions lead to steady-state gamma distributions for wealth. The mathematical derivation (Eq. (11)) shows how the shape factor ν depends on the mean and variance of the preference distribution and the variance of successive price ratios. The latter is clearly determined by the preference distribution, but we do not, at present, have a method other than simulation to calculate it. While theory predicts the asymptotic behavior for large N, it is remarkable that simulations with as few as 1000 individuals led to equilibrium distributions very close to the asymptotic limit. Thus, characteristics of the "thermodynamic limit" may already be apparent in systems with modest numbers of individuals.

We do not propose the model outlined here as an accurate description of human behavior in economic markets. Rather, we view the model as an aid to thinking about how statistical phenomena may influence economic outcomes such as the distribution of wealth. In the model, inequality in wealth is seen to be a consequence of stochastic forces even when all individuals are equivalent. Wealth disparity increases, the greater the diversity in preferences, and decreases, the larger the number of goods traded. Since the shape parameter ν of the gamma distribution corresponds to the number of goods traded in the case of uniform preferences, nonuniform preference functions can be viewed as having different *effective* numbers of goods traded, with more heterogeneous (U-shaped) preferences corresponding to a reduced number of goods. The similarities between the economics model and the statistical treatment of an ideal gas in physics suggests ways in which the economics model might be modified to take into account interactions between individuals that allow correlation among their preferences. The method used to derive probabilistically the form of the steady state depends on the particular feature of the transition mechanism that individual agents' wealths evolve independently and symmetrically except for rebalancing in each generation. Such a rebalancing mechanism seems characteristic of price-allocation within mathematical economics but can arise also in neural-network or machine-learning models.

REFERENCES

1. L. B. Bachelier, "Théorie de la spéculation," Gauthier-Villars, Paris, 1900.
2. P. Samuelson, Proof that properly anticipated prices fluctuate randomly, *Indust. Manag. Rev.* **6** (Spring), 41–49 (1965).
3. A. Einstein, "Investigations on the Theory of Brownian Movement" (R. Fürth, Ed., A. D. Cowper, transl.), Dover, New York 1956.
4. A. I. Khinchin, "Mathematical Foundations of Statistical Mechanics," Dover, New York, 1949.
5. D. Ruelle, "Statistical Mechanics: Rigorous Results," Benjamin, Reading, MA, 1969.
6. B. Mandelbrot, The variation of certain speculative prices, *J. Business* **36** (1963), 392–417.
7. E. Fama, The behavior of stock-market prices, *J. Business* **37** (1965), 226–241.
8. R. N. Mantegna and H. E. Stanley, Scaling behavior in the dynamics of an economic index, *Nature* **376** (1996), 46–49.
9. T. Lux, The socio-economic dynamics of speculative markets: interacting agents, chaos, and the fat tails of return distributions, *J. Econ. Behav. Organ.* **33** (1998), 143–165.
10. R. H. Day and W. Huang, Bulls, bears, and market sheep, *J. Econ. Behav. Organ.* **14** (1990), 299–329.
11. P. Bak, M. Paczuski, and M. Shubik, Price variations in a stock market with many agents, *Physica A* **246** (1997), 430–453.
12. N. Georgescu-Rogen, "The Entropy Law and the Economic Process," Harvard Univ. Press, Cambridge, MA, 1971.
13. E. Farjoum and M. Machover, "Laws of Chaos: A Probabilistic Approach to Political Economy," Verso, London, 1983.
14. D. K. Foley, A statistical equilibrium theory of markets, *J. Econ. Theory* **62** (1994), 321–345.
15. R. Mantegna and H. Stanley, "An Introduction to Econophysics," Cambridge Univ. Press, Cambridge, UK, 2000.