No books or notes allowed. No laptop or wireless devices allowed. Write clearly.

Name: ____________________

Question:	1	2	3	Total
Points:	35	30	35	100
Score:				

Question 1 .... *35 point*

Let T_1 and T_2 be two independent exponential r.v. with $E(T_1) = E(T_2) = 1$ and let N be a Poisson r.v. with $E(N) = 1$.

(a) (10 points) Show that $P(T_1 < 1) = P(N \geq 1)$.

Solution: The p.d.f. of T_1 is

$$f_1(t_1) = e^{-t_1}.$$

We have

$$P(T < 1) = \int_0^1 e^{-t_1} dt_1 = 1 - e^{-1}$$

while

$$P(N \geq 1) = 1 - P(N = 0) = 1 - e^{-1}.$$

(b) (15 points) Compute the probability that $T_1 + T_2 < 1$, *i.e.* $P(T_1 + T_2 < 1)$. (**Hint**: compute the joint p.d.f. $f(t_1, t_2)$ of T_1 and T_2 and evaluate $\int_A f(t_1, t_2) dt_1 dt_2$ on the proper set A.)

Solution: Since the p.d.f. of T_2 is

$$f_2(t_2) = e^{-t_2}$$

and T_1 and T_2 are independent we get

$$f(t_1, t_2) = e^{-(t_1+t_2)}.$$

We get

$$P(T_1 + T_2 < 1) = \iint_{t_1+t_2<1} e^{-(t_1+t_2)} dt_1 dt_2 = \int_0^1 \int_0^{1-t_1} e^{-(t_1+t_2)} dt_2 dt_1 =$$
$$\int_0^1 e^{-t_1}(1 - e^{-(1-t_1)}) dt_1 = \int_0^1 e^{-t_1} dt_1 - e^1 = 1 - 2e^{-1}.$$

(c) (10 points) Show that $P(T_1 + T_2 < 1) = P(N \geq 2)$.

Solution: We have

$$P(N \geq 2) = 1 - P(N = 0) - P(N = 1) = 1 - 2e^{-1}.$$

Question 2 . . . . *30 point*

(a) (15 points) Let X_1 and X_2 be two independent r.v. with $X_1 \simeq \mathcal{N}(1,9)$ and $X_2 \simeq \mathcal{N}(-3,16)$. Calling $Y = X_1 - X_2$, compute the probability that $0 < Y < 10$, that is $P(0 < Y < 10)$.

Solution: We have that Y is normal since it is the sum of two independent normal r.v.. Moreover $E(Y) = E(X_1) - E(X_2) = 4$ while $V(Y) = V(X_1) + V(X_2) = 25$. We thus have that $Y \simeq \mathcal{N}(4,25)$ so that

$$P(0 < Y < 10) = P\left(\frac{0-4}{5} < \frac{Y-4}{5} < \frac{10-4}{5}\right) = \Phi(1.2) - \Phi(-0.8) =$$
$$= 0.8849 - 0.2119 = 0.6730\,.$$

(b) (15 points) Assume now that that X_1 and X_2 are as in point a) but for the fact that $\rho_{X_1,X_2} = 2/3$. Assume moreover that $Y = X_1 - X_2$ is still a normal r.v.. Compute the probability that $0 < Y < 10$, that is $P(0 < Y < 10)$.

Solution: In this case we still have that $E(Y) = E(X_1) - E(X_2) = 4$ while $V(Y) = V(X_1) + V(X_2) - 2\mathrm{Cov}(X_1, X_2)$. Since $\mathrm{Cov}(X_1, X_2) = \rho_{X_1,X_2}\sigma_{X_1}\sigma_{X_2}$ we have that $Y \simeq \mathcal{N}(4,9)$ and

$$P(0 < Y < 10) = P\left(\frac{0-4}{3} < \frac{Y-4}{5} < \frac{10-4}{3}\right) = \Phi(2) - \Phi(-1.33) =$$
$$= 0.9772 - 0.0918 = 0.8854\,.$$

Question 3 .... *35 point*

You decide to go to Las Vegas to play roulette. You select a random number between 0 and 36 and bet \$1 on that number. The outcome of the roulette is a number between 0 and 36, *i.e.* there are 37 possible outcomes. They all have the same probability. If the outcome is equal to the number you selected, you get back \$36, that is you win \$35. If not you get back \$0, that is you lose your dollar. Let X be the r.v. that describe the amount you win (or lose).

(a) (15 points) Compute the p.m.f. of X, its expected value $E(X)$ and its variance $V(X)$.

Solution: There are only two possible value for X, either 35 or -1. we get

$$p(35) = \frac{1}{37} \qquad p(-1) = \frac{36}{37}.$$

Thus we have

$$\mu = E(X) = -\frac{1}{37} = -0.027 \qquad \sigma^2 = \left(35^2\frac{1}{37} + \frac{36}{37}\right) - \frac{1}{37^2} = 34.08$$

(b) (20 points) Suppose that in that evening you play 300 times in the same way. Let S_X be the total amount of money you win (or loose) during the evening. Using the C.L.T., compute the expected value $E(S_X)$ of S_X, its variance $V(S_X)$ and the probability that you will not loose money, that is $P(S_X > 0)$.

Solution: Clearly

$$E(S_X) = 300\mu \qquad\qquad V(S_X) = 300\sigma^2$$

Using the C.L.T. we get that

$$S_X \sim \mathcal{N}(300\mu, 300\sigma^2)$$

so that we have

$$\begin{aligned} P(S_X > 0) =& P\left(\frac{S_X - 300\mu}{\sqrt{300}\sigma} > -\frac{300\mu}{\sqrt{300}\sigma}\right) = \Phi\left(\sqrt{300}\frac{\mu}{\sigma}\right) = \\ =& \Phi(-0.08) = 0.47 \end{aligned}$$

Useful Facts

- **Exponential Distribution**: if T is a exponential r.v. with parameter λ then its p.d.f. is
$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if} \quad t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
and $E(t) = 1/\lambda$.

- **Poisson Distribution**: if N is a Poisson r.v. with parameter μ then its p.m.f. is
$$p(n) = \frac{\mu^n}{n!} e^{-\mu}$$
and $E(N) = \mu$.

- **Variance and Covariance**: If X_1 and X_2 are r.v. and a_1 and a_2 real numbers, we have
$$V(a_1 X_1 + a_2 X_2) = a_1^2 V(X_1) + a_2^2 V(X_2) + 2 a_1 a_2 \mathrm{Cov}(X_1, X_2)\,.$$

- **Correlation**: If X_1 and X_2 are r.v then
$$\rho_{X_1, X_2} = \frac{\mathrm{Cov}(X_1, X_2)}{\sigma_{X_1} \sigma_{X_2}}\,.$$

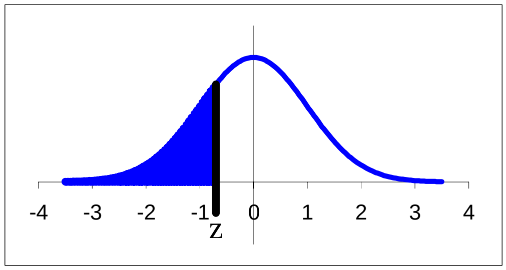
- **C.L.T.**: Let X_i, $i = 1, \ldots, N$, be N independent and identically distributed r.v. Then $T_0 = \sum_{i=1}^{N} X_i$ is a Normal r.v.

Table 1a: Standard Normal Probabilities

The values in the table below are cumulative probabilities for the standard normal distribution Z (that is, the normal distribution with mean 0 and standard deviation 1). These probabilities are values of the following integral:

$$P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Geometrically, the values represent the area to the left of z under the density curve of the standard normal distribution:



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641