No books or notes allowed. No laptop or wireless devices allowed. Write clearly.

Name: ______________________________

Question:	1	2	3	4	Total
Points:	30	30	20	30	110
Score:					

Question 1 .... *30 point*

The following numbers x_i, $i = 1, \ldots, 19$, represent a sample of size $n = 19$ from a given population.

26.7	28.4	23.4	27.8	26.8
20.1	26	23.9	29.2	20
24.6	24.2	24.6	27.7	23.2
25.5	24.7	20.4	13.2	

(a) (10 points) Compute the sample median and fourth spread.

Solution:

After ordering the data you obtain

13.2	20	20.1	20.4	23.2
23.4	23.9	24.2	24.6	24.6
24.7	25.5	26	26.7	26.8
27.7	27.8	28.4	29.2	

so that:

$\tilde{x} = 24.6$

$lf = (23.2 + 23.4)/2 = 23.3 \qquad uf = (26.7 + 26.8)/2 = 26.75$

$fs = 26.75 - 23.3 = 3.45$

(b) (10 points) Knowing that $\sum_{i=1}^{19} x_i = 460.4$ and $\sum_{i=1}^{19} x_i^2 = 11414$ compute the sample mean and variance.

Solution:

$$\bar{x} = \frac{460.4}{19} = 24.23$$

$$s^2 = \frac{1}{18}\left(11414 - \frac{460.4^2}{19}\right) = 14.33$$

(c) (10 points) After checking for outlier, draw a box plot of the data.

Solution: Observe that $13.2 < 23.3 - 1.5 \cdot 3.45$ while $13.2 > 23.3 - 3 \cdot 3.45$ so that 13.2 is the only outlier of the sample.

Question 2 . . . . *30 point*

Let $\mathcal{S}$ be a sample space and $A, B, C \subset \mathcal{S}$ be three events. Assume that A and B are independent and that A and C are independent. Assume moreover that $P(A) = 0.5$, $P(B) = P(C) = 0.3$. Use these assumptions to solve points (a), (b) and (c).

(a) (10 points) Show that A and B' are independent.

Solution:
Since $(A \cap B) \cup (A \cap B') = A$

$$P(A \cap B') + P(A \cap B) = P(A)$$

Since $P(A \cap B) = P(A)P(B)$ we have

$$P(A \cap B') = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B')$$

(b) (10 points) If $B \cap C = \emptyset$, compute $P(A \cap (B \cup C))$. (*Hint*: use that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$)

Solution:
Observe that $(A \cap B) \cap (A \cap C) = \emptyset$. Thus

$$\begin{aligned} P(A \cap (B \cup C)) &= P((A \cap B) \cup (A \cap C)) = P(A \cap B) + P(A \cap C) = \\ &= P(A)P(B) + P(A)P(C) = 0.5 \cdot 0.6 \end{aligned}$$

(c) (10 points) Show that if A is independent from $B \cap C$ then A is independent from $B \cup C$.

Solution: As before $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ so that

$$\begin{aligned} P(A \cap (B \cup C)) &= P((A \cap B) \cup (A \cap C)) = \\ &= P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C)) \end{aligned}$$

But $(A \cap B) \cap (A \cap C) = A \cap (B \cap C)$ so that $P((A \cap B) \cap (A \cap C)) = P(A)P(B \cap C)$. Finally

$$P(A \cap (B \cup C)) = P(A)\,(P(B) + P(C) - P(B \cap C)) = P(A)P(B \cup C)$$

Question 3 .... *20 point*

Let X_1 and X_2 be two r.v that can take the values -1, 0 and 1. Assume that each of the three value has the same probability for both X_1 and X_2 and that the events $\{X_1 = i\}$ and $\{X_2 = j\}$ are independent for every i and j, $i, j = -1, 0, 1$.

(a) (10 points) Compute the p.m.f. of $Y = X_1 + X_2$ and $Z = X_1 X_2$.

Solution: Observe that Y can take the values $-2, -1, 0, 1, 2$. There is one outcome each for which $Y = -2$ or $Y = 2$, two outcomes each for $Y = -1$ or $Y = 1$ and three outcomes for $Y = 0$ so that:

$$p_Y(-2) = p_Y(2) = \frac{1}{9} \qquad p_Y(-1) = p_Y(1) = \frac{2}{9} \qquad p_Y(0) = \frac{1}{3}$$

On the other hand, Z can take the values $-1, 0, 1$. Two outcomes give -1 or 1 and five outcomes give 0. Thus

$$p_Z(-1) = p_Z(1) = \frac{2}{9} \qquad p_Z(0) = \frac{5}{9}$$

(b) (10 points) Compute $E(Y)$, $E(Z)$ and $V(Y)$, $V(Z)$.

Solution:

$$E(Y) = -2 \cdot p_Y(-2) - 1 \cdot p_Y(-1) + 0 \cdot p_Y(0) + 1 \cdot p_Y(1) + 2 \cdot p_Y(2) = 0$$
$$E(Z) = -1 \cdot p_Z(-1) + 0 \cdot p_Z(0) + 1 \cdot p_Z(1) = 0$$

While

$$E(Y^2) = 4 \cdot p_Y(-2) + 1 \cdot p_Y(-1) + 0 \cdot p_Y(0) + 1 \cdot p_Y(1) + 4 \cdot p_Y(2) = \frac{4}{3}$$
$$E(Z^2) = 1 \cdot p_Z(-1) + 0 \cdot p_Z(0) + 1 \cdot p_Z(1) = \frac{4}{9}$$

so that

$$V(Y) = E(Y^2) - E(Y)^2 = \frac{4}{3}$$
$$V(Z) = E(Z^2) - E(Z)^2 = \frac{4}{9}$$

Question 4 . . . . *30 point*

A factory produces 1000 computers. Each computer has a probability $p = 0.01$ to have an internal defect. The factory has a quality control department that is able to detect and discard a defected computer with probability 1. It also has a probability $s = 0.005$ of discarding a working computer.

(a) (10 points) Let X be the number of defected computer among the 1000 produced. Write the p.m.f. of X and compute $E(X)$ and σ_X.

Solution: Clearly X is a Binomial r.v. with parameter 1000 and 0.01. It follows that

$$p_x(x) = P(X = x) = \binom{1000}{x} 0.01^x 0.99^{1000-x}$$

Moreover

$$E(X) = 1000 \cdot 0.01 = 10 \qquad \sigma_X = \sqrt{V(X)} = \sqrt{1000 \cdot 0.01 \cdot 0.99} = \sqrt{9.9}$$

(b) (10 points) Compute the probability that a randomly selected computer will be discarded by the quality control department and the probability that a discarded computer is actually defected.

Solution:
Call A the event $\{$*computer is defected*$\}$ and B the event $\{$*computer is discarded*$\}$. Then

$$P(B) = P(B|A)P(A) + P(B|A')P(A') = 1 \cdot 0.01 + 0.005 \cdot 0.99 = 0.015$$

while

$$P(A|B) = P(B|A)\frac{P(A)}{P(B)} = 1\frac{0.01}{0.015} = 0.67$$

(c) (10 points) Call Y the r.v. that describes the number of non discarded computers and Z the number of discarded computers that are working. Write the p.m.f. of Y and Z.

Solution:

Clearly Y and Z are both Binomial r.v.. Y has parameters 1000 and $(1 - 0.015) = 0.985$. Z has parameter 1000 and p is the probability that a computer is both working and discarded. This last probability is the probability that the computer is working time the probability that it will be discarded given that it is working. Thus $p = 0.99 * 0.005 = 0.00495$.

It follows that

$$
\begin{aligned}
p_Y(y) &= \binom{1000}{y} 0.985^y 0.015^{1000-y} \\
p_Z(z) &= \binom{1000}{z} 0.0495^z (1 - 0.0495)^{1000-z}
\end{aligned}
$$