No books or notes allowed. No laptop or wireless devices allowed. Write clearly.

Name: ______________________________

Question:	1	2	3	4	5	Total
Points:	20	20	20	10	40	110
Score:						

Question 1 . . . . *20 point*

In a bucket there are 6 black balls, 4 green and 4 red. Balls of the same color are identical.

(a) (10 points) In how many different ways can you order these balls?

Solution: If the ball are considered different there would be 14! possible way to oder them. The number requested is thus

$$\frac{14!}{6!\,4!\,4!}$$

(b) (10 points) If you are required to order them in such a way that every red ball is followed by a green one, in how many different ways can you order them?

Solution: You can now consider a sequence red followed by green as a single object so that the number requested is

$$\frac{10!}{6!\,4!}$$

Question 2 .... *20 point*

In a stock of 1000 bulbs there are 500 bulb of type 1 and 500 of type 2. You know that the lifetime of a bulb of type 1 is described by an exponential r.v. with parameter $l = 1$ while the lifetime of a bulb of type 2 is described by an exponential r.v. with parameter $l = 2$. You take one bulb at random from the stock. Remember that an exponential r.v. with parameter $l$is described by a p.d.f $f(t) = l\exp(-lt)$.

(a) (10 points) Compute the probability that the bulb will work longer than t.(**Hint**: let T the lifetime of the bulb, $A_1 = \{$*selected bulb is of type 1*$\}$ and $A_2 = \{$*selected bulb is of type 2*$\}$. The requested probability is $P(T > t)$. Write it in term of $P(T > t|A_1)$ and $P(T > t|A_2)$.)

Solution: Form the law of total probability we have

$$P(T > t) = P(T > t|A_1)P(A_1) + P(T > t|A_2)P(A_2)$$

Since a bulb of type 1 has an exponentially distributed life time with $l = 1$ we have

$$P(T > t|A_1) = e^{-t}$$

while

$$P(T > t|A_1) = e^{-2t}$$

Finally, since there is an equal number of bulbs of type 1 or 2, we have

$$P(A_1) = P(A_2) = \frac{1}{2}$$

Thus we get

$$P(T > t) = \frac{1}{2}\left(e^{-t} + e^{-2t}\right)$$

(b) (10 points) Compute the p.d.f of T.

Solution: From point a) it follows that the c.d.f of T is

$$F(t) = 1 - \frac{1}{2}\left(e^{-t} + e^{-2t}\right)$$

so that the p.d.f. is

$$f(t) = \frac{d}{dt}F(t) = \frac{1}{2}\left(e^{-t} + 2e^{-2t}\right)$$

Question 3 .... *20 point*

The temperature (in Fahrenheits) in Atlanta in a November day is described by a normal random variable X with expected value 60 and standard deviation 10.

(a) (10 points) Find the value c such that the probability that the temperature is between $60 - c$ and $60 + c$ is 0.95.

Solution: Let $Z = (X - 60)/10$. We have

$$\begin{aligned} P(60 - c < X < 60 + c) &= P\left(-\frac{c}{10} < Z < \frac{c}{10}\right) = \\ &= 1 - 2P\left(Z < -\frac{c}{10}\right) = 1 - 2\Phi\left(-\frac{c}{10}\right) \end{aligned}$$

We need to find z such that

$$\Phi(z) = \frac{1 - 0.95}{2} = 0.025$$

From the table we get $z = -1.96$ so that

$$-\frac{c}{10} = -1.96$$

or

$$c = 19.6$$

(b) (10 points) Let Y be the temperature in Centigrade. What is the expected value and standard deviation of Y? (remember that x Fahrenheit are equivalent to $y = 5(x - 32)/9$ Centigrade).

Solution: We have

$$E(Y) = \frac{5(E(X) - 32)}{9} = \frac{5 \cdot 28}{9}$$

while

$$V(Y) = \frac{25}{81}V(X) = \frac{2500}{81} \qquad \sigma_Y = \frac{5}{9}\sigma_X = \frac{50}{9}$$

Question 4 . . . . *10 point*

Consider the following six observations on bearing lifetime (in hours):

$$9.1677 \quad 9.9044 \quad 10.2944 \quad 8.6638 \quad 10.7143 \quad 11.6236.$$

Construct a normal probability plot and comment on the plausibility of the normal distribution as a model for bearing lifetime.

Solution: To construct a probability plot we need to know the 6 value z_i such that

$$\Phi(z_i) = \frac{2i-1}{2}$$

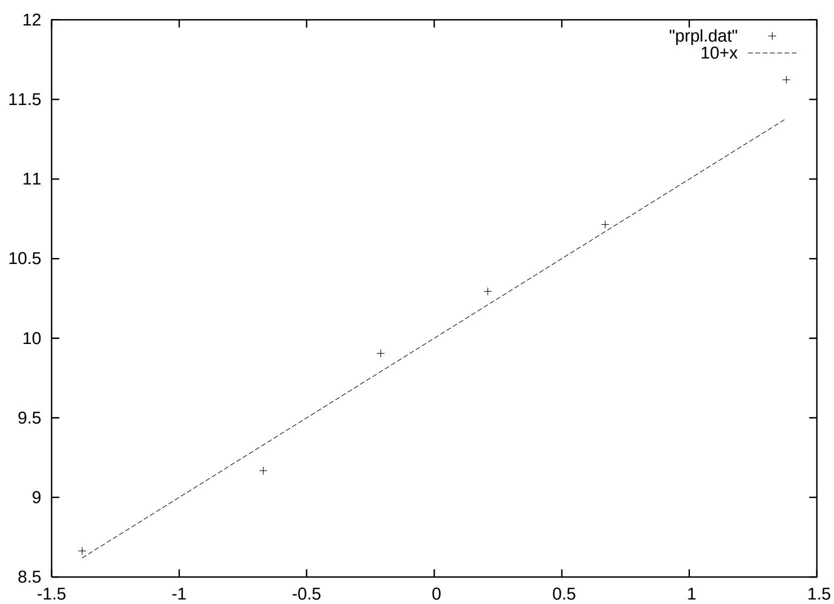
for $i = 1, \ldots, 6$. They are

$$z_1 = -1.38 \quad z_2 = -0.67 \quad z_3 = -0.21 \quad z_4 = 0.21 \quad z_5 = 0.67 \quad z_6 = 1.38$$

the plot is thus

z	-1.38	-0.67	-0.21	0.21	0.67	1.38
x	8.6638	9.1677	9.9044	10.2944	10.7143	11.6236

that gives the graph below. The data appear to be normal witn expected value 10 and standard deviation 1.



Question 5 .... *40 point*

The results of an experiment is described by a r.v. X uniformly distributed between 0 and 1. Consider the r.v. P given by

$$P = \begin{cases} 1 & \text{if} \quad X \geq 0.5 \\ 0 & \text{if} \quad X < 0.5 \end{cases}$$

and the r.v. Q given by

$$Q = \begin{cases} 0 & \text{if} \quad X \geq 0.5 \\ 1 & \text{if} \quad X < 0.5 \end{cases}$$

(a) (10 points) Compute the joint p.m.f. of P and Q. Compute the marginal p.m.f. of P and of Q.

Solution: If $P = 1$ than $Q = 0$ and viceversa so that, if $p(x, y)$ is the joint p.d.f. we have:

$$p(0,0) = p(1,1) = 0 \qquad p(0,1) = p(1,0) = 0.5$$

So we get

$$p_P(0) = p_P(1) = 0.5 \qquad p_Q(0) = p_Q(1) = 0.5$$

(b) (10 points) Compute $\operatorname{corr}(P, Q)$.

Solution: Short way: observe that $P + Q \equiv 1$ so that $P = 1 - Q$. This means that

$$\operatorname{corr}(P, Q) = -1.$$

Long way:

$$\begin{array}{lll} E(P) = 0.5 & E(Q) = 0.5 & E(PQ) = 0 \\ E(P^2) = 0.5 & E(Q^2) = 0.5 & \\ V(P) = 0.25 & V(Q) = 0.25 & \end{array} \tag{1}$$

so that

$$\operatorname{corr}(P, Q) = \frac{E(PQ) - E(P)E(Q)}{\sqrt{V(P)V(Q)}} = -1$$

(c) (10 points) Suppose now you repeat the experiment 100 times obtaining 100 independent and identically distributed r.v. X_i uniformly distributed between 0 and 1. For every X_i, define P_i and Q_i as in point (a) above. Let

$$\bar{X} = \frac{1}{100}\sum_{i=1}^{100} X_i$$
$$\bar{P} = \frac{1}{100}\sum_{i=1}^{100} P_i$$
$$\bar{Q} = \frac{1}{100}\sum_{i=1}^{100} Q_i$$

Give the approximate p.d.f of $\bar{X}$, $\bar{P}$ and $\bar{Q}$.

Solution: We have that

$$E(X_i) = 0.5 \qquad V(X_i) = E(X_i^2) - E(X_i)^2 = \frac{1}{12}$$

So, using the CLT we get

$$X \simeq \mathcal{N}\left(0.5, \frac{1}{1200}\right)$$

while

$$P \simeq \mathcal{N}\left(0.5, \frac{1}{400}\right) \qquad Q \simeq \mathcal{N}\left(0.5, \frac{1}{400}\right)$$

(d) (10 points) Compute $\operatorname{corr}(\bar{P}, \bar{Q})$.

Solution: We still have that $\bar{P} = 1 - \bar{Q}$ so that

$$\operatorname{corr}(\bar{P}, \bar{Q}) = -1$$