Spring 04 | Name: ____________________
Math 3770 | Second Midterm | Bonetto

1) Let X_1 and X_2 be two independent normal standard r.v.. Let

$$Y_1 = X_1 + X_2$$
$$Y_2 = X_1 - X_2$$

Compute

a) The expected value and variance of Y_1 and Y_2.

b) $\mathrm{Cov}(Y_1, Y_2)$ and $\mathrm{Corr}(Y_1, Y_2)$.

c) Can Y_1 and Y_2 be independent?

We know that $E(a_1X_1 + a_2X_2) = a_1E(X_1) + a_2E(X_2)$ while $V(a_1X_1 + a_2X_2) = a_1^2V(X_1) + a_2^2V(X_2)$ so that for Y_1 we have

$$E(Y_1) = E(X_1) + E(X_2) = 0 \qquad V(Y_1) = V(X_1) + V(X_2) = 2$$

while for Y_2:

$$E(Y_2) = E(X_1) + E(X_2) = 0 \qquad V(Y_2) = V(X_1) + V(X_2) = 2.$$

For the covariance we have:

$$\mathrm{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = E\left((X_1 + X_2)(X_1 - X_2)\right) = E(X_1^2) - E(X_2^2) = 0$$

while

$$\mathrm{Corr}(Y_1, Y_2) = \frac{\mathrm{Cov}(Y_1, Y_2)}{\sigma_{X_1}\sigma_{X_2}} = 0$$

Finally since $\mathrm{Cov}(Y_1, Y_2) = 0$ it is possible that Y_1 and Y_2 are independent.

2) A group of 600 students this semester attempted the exams for Calculus II (CII) and Linear Algebra (LA). Assume that the possible grades are just 0,1 or 2. The combined results of the exam are given in the following table:

		CII		
		2	1	0
	2	200	80	20
LA	1	20	100	80
	0	80	15	5

Let X_1 be the r.v. for the result of CII and X_2 the r.v. for the result of LA. Compute:

a) the marginal p.m.f. of X_1 and X_2.

b) the expected value of X_1, X_2 and $X_1 + X_2$.

c) if $f_{X_1}(x_1|x_2)$ is the conditional p.m.f. of X_1 given X_2, compute $f_{X_1}(2|2)$.

d) are X_1 and X_2 independent?

Let us call $f(x_1, x_2)$ the joint p.m.f. of X_1 and X_2. The marginal are given by:

$$f_{X_1}(2) = \frac{300}{600} = 0.5 \quad f_{X_1}(1) = \frac{195}{600} = 0.325 \quad f_{X_1}(0) = \frac{105}{600} = 0.175$$
$$f_{X_2}(2) = \frac{300}{600} = 0.5 \quad f_{X_1}(1) = \frac{200}{600} = 0.333 \quad f_{X_1}(0) = \frac{100}{600} = 0.167$$

The expected values are given by:

$$E(X_1) = 0 \cdot 0.175 + 1 \cdot 0.325 + 2 \cdot 0.5 = 1.325$$
$$E(X_2) = 0 \cdot 0.167 + 1 \cdot 0.333 + 2 \cdot 0.5 = 1.333$$
$$E(X_1 + X_2) = E(X_1) + E(X_2) = 2.658$$

For the conditional probability we have:

$$f_{X_1}(2|2) = \frac{f(2,2)}{f_{X_2}(2)} = \frac{0.333}{0.5} = 0.677$$

Finally observe that $f(2,2) = 0.333$ is different from $f_{X_1}(2)f_{X_2}(2) = 0.25$ so that X_1 and X_2 are not independent.

3) A passenger can take two different bus lines to go to work. Both lines stop at the same bus stop. Let T_1 the r.v. giving the time of arrival of the bus of line 1 and T_2 the r.v. giving the time of arrival of the bus of line 2. You know that T_1 is exponential with parameter 10 and T_2 is exponential with parameter 20 when time is measured in minutes. Let T be the random variable giving the time our passenger will wait for a bus i.e $T = \min(T_1, T_2)$ the smallest between T_1 and T_2.
   a) compute $P(T > t)$ i.e. the probability that after t minutes no bus as arrived jet?
   b) What is the p.d.f. of T? (**Hint**: compute the c.d.f and remember that the p.d.f is the derivative of the c.d.f.)
   c) What is the average time the passenger will wait at the bus stop?

The probability that after t minutes no bus as arrived jet is the probability that both T_1 and T_2 are larger than t. But T_1 and T_2 are independent so that:

$$P(T > t) = P(T_1 > t \text{ and } T_2 > t) = P(T_1 > t)P(T_2 > t) = e^{-10t}e^{-20t} = e^{-30t}$$

The c.d.f of T is

$$F(t) = P(T \leq t) = 1 - P(T > t) = 1 - e^{-30t}$$

so that differentiating we get that the p.d.f. of T is $f(t) = 30e^{-30t}$. In particular T is exponential with parameter 30. Clearly the expected value of T is $E(T) = 1/30 = 0.0333$ and it is the average time the passenger will wait.