1. There are two stoks on the market, stock A and stock B. Every day the price of stock A will either increase by 1 dollar with probability 0.6 or decrease by one dollar with probability 0.4. Similarly stock B will either increase by 2 dollars with probability 0.55 or decrease by two dollars with probability 0.45.

   a) If, at the beginning, the prive of stock A was 100 dollars and that of stock B was 120 dollars, what is the (approximate) price distribution of stock A and B after 100 days?

      Due to the CLT the distribution of prices will be normal. The average variation of stock in a single day is $E(X_A) = 0.2$ *while the variance is* $V(X_A) = 0.96$. *It follows that the price of A after 100 days* Y_A *is normal with average* $\mu_A = 120$ *and variance* $\sigma_A^2 = 96$. *In the same way the price of stock B after 100 days* Y_B *will be normal with average* $\mu_B = 140$ *and variance* $\sigma_B^2 = 396$.

   b) Compute the probability that, after 100 days, the price of stock A will be less than than its initial price. Do the same for stock B.

      We need

$$P(Y_A < 100) = P\left(\frac{Y_A - 120}{\sqrt{96}} < \frac{100 - 120}{\sqrt{96}}\right) = \Phi(-2.04) = 0.0207$$

      and

$$P(Y_B < 120) = P\left(\frac{Y_B - 140}{\sqrt{396}} < \frac{100 - 140}{\sqrt{396}}\right) = \Phi(-1.00) = 0.1587$$

   c) At the beginning you have 20000 dollars and you buy 140 stock A and 50 stock B. What is the probability that after 100 days you have lost money? Doubled your initial capital?

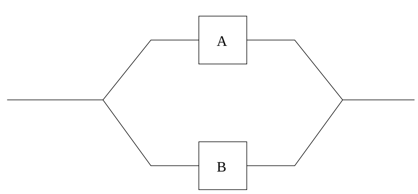
      The value of your stok after 100 days will be $Y = 140 Y_A + 50 Y_B$. *This is a normal r.v. with average* $\mu = 140 \cdot 120 + 50 \cdot 140 = 23800$ *and varinacs* $\sigma^2 = 140^2 \cdot 96 + 50 \cdot 396 = 2871600$. *So that we have*

$$P(Y < 20000) = P\left(\frac{Y - 23800}{\sqrt{2871600}} < \frac{20000 - 23800}{\sqrt{2871600}}\right) = \Phi(-2.24) = 0.0125$$

      and

$$P(Y > 40000) = P\left(\frac{Y - 40000}{\sqrt{2871600}} < \frac{20000 - 40000}{\sqrt{2871600}}\right) = \Phi(11.80) \sim 0$$

2. A circuit is composed by two devices, A and B, connected in parallel (see figure).



   You know that the circuit fails if both devices fail and that each device has a exponential life time with parameter λ, this means that if T_A and T_B are the r.v. describing the lifetime of device A and B respectively then both T_A and T_B are exponential r.v. with parameter λ.

   a) Find the p.d.f. of the lifetime T of the circuit.

The c.d.f. of T is given by

$$P(T<t)=P(T_A<t\,\&\,T_B<t)=P(T_A<t)P(T_B<t)=(1-e^{-\lambda t})^2$$

so that the p.d.f. is

$$f(t)=2\lambda e^{-\lambda t}(1-e^{-\lambda t}).$$

You observe the lifetime of 10 such circuits and obtain the following breacking-down times:

3.47 1.69 1.78 0.45 1.60 3.91 2.81 7.66 0.72 2.32

A) Use the method of moments to give an estimate of λ.

We have

$$E(T)=\int_0^\infty t2\lambda e^{-\lambda t}(1-e^{-\lambda t})\mathrm{dt}=\frac{2}{\lambda}-\frac{1}{2\lambda}=\frac{3}{2\lambda}$$

Hence we have

$$\hat{\lambda}=\frac{3}{2\bar{t}}=\frac{3}{2\cdot 2.64}=0.61$$

B) Use the method of maximum likelihood to estimate λ.

C) The half live of the circuit is defined as that time when, starting with a big set of circuit, only half of them are still working. Give an estimate of the half life of the circuit using the invariance principle.

Let's call λ_{ML} the maximun likelihood estimator. We have that the half life $t_{0.5}$ is given by

$$P(T<t_{05})=(1-e^{-\lambda t_{0.5}})^2=0.5$$

so that

$$\hat{t}_{0.5}=-\ln\left(1-\sqrt{0.5}\right)\lambda_{\mathrm{ML}}^{-1}$$

3. **(Bonus)** The following data come from a population uniformly distributed between $-A$ and A:

-0.255 0.915 -0.185 -0.363 0.508 -0.559 1.436 1.013 1.251 -0.539

a) Use the method of moment to estimate A.

We have that $E(X)=0$ so that we have to look for the second moment. We have $E(X^2)=A^2/3$. So that we can find

$$\hat{A}=\sqrt{\frac{3}{10}\sum_{i=1}^{10}x_i^2}=1.404$$

b) Use maximum likelihood to estimate A.

The p.d.f. of X is $f(x)=1/A$ if $|x|<A$ and 0 otherwise. So that the likelihood function is $h(x)=1/A^{10}$ if $\max(x_i)<A$ and 0 otherwise. It follows that

$$A_{\mathrm{ML}}=\max(|x_i|)=1.436$$

4. Among the sudent that attempted the exams of Calculus III (CIII) and Differential Equation (DE) is oberved that the joint probability of passing or failing the exams is given by the following table:

		CIII f	p
DE	f	0.3	0.1
	p	0.1	0.5

This means, for example, that the probability for a student to pass both exams is 0.5 while the probability of passing DE and failing CIII is 0.1. Given a student, let X be the r.v. that describes his result in DE and Y the r.v. that describes his result in CIII. Both variables take value 0 or 1 where 0 mean fail and 1 pass.

a) Compute the marginals $f_X(x)$, $f_Y(y)$.

We have:

$$f_X(0) = 0.3 + 0.1 = 0.4 \qquad f_X(1) = 0.6$$

and

$$f_Y(0) = 0.4 \qquad f_Y(1) = 0.6$$

b) Compute the conditional p.d.f. $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

From

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

we get

$$f_{X|Y}(0|0) = 0.3/0.4 = 0.75 \qquad f_{X|Y}(1|0) = 0.1/0.4 = 0.25$$
$$f_{X|Y}(0|1) = 0.1/0.6 = 0.167 \qquad f_{X|Y}(1|1) = 0.5/0.6 = 0.75$$

and

$$f_{X|Y}(0|0) = 0.3/0.4 = 0.75 \qquad f_{X|Y}(1|0) = 0.1/0.4 = 0.25$$
$$f_{X|Y}(0|1) = 0.1/0.6 = 0.167 \qquad f_{X|Y}(1|0) = 0.6/0.6 = 0.75.$$

c) Compute Corr(X, Y).

We have $E(X) = E(Y) = 0.6$, $E(X^2) = E(Y^2) = 0.6$ *so that* $V(X) = V(Y) = 0.6 - 0.36 = 0.34$. *Finally* $E(\mathrm{XY}) = 0.5$ *so that*

$$\mathrm{Corr}(X, Y) = \frac{0.5 - 0.6 \cdot 0.6}{\sqrt{0.34^2}} = 0.411$$

d) Suppose now that you choose 3 students at random and let N_X the number of students among the 3 that passed DE and N_Y the number of students among the 3 that passed CII. Compute the probability that $N_X = 1$ and $N_Y = 2$.

$N_X = 1$ *means that one student passed DE and the other two failed while* $N_Y = 2$ *means that two students passed CIII and one failed. There are two possibilities:*

i. the student that passed DE failed CIII, in which case the other two student passed CIII and failed DE.

ii. *the student that passed DE passed CIII, in which case one of the remaining students failed both and the other passed CIII but failed DE.*

In the first case there are 3 possible chioces of the student that passed DE. Each chioce has probability $p_1 = 0.1^3$=*0.001. In the second case we have 6 chioces, because all students are different. Each choice has probability* $p_2 = 0.5 \cdot 0.3 \cdot 0.1 = 0.015$. *Finally the probability is*

$$P(N_X = 1 \,\&\, N_Y = 2) = 4 \cdot 0.001 + 6 \cdot 0.015 = 0.094.$$