Name: ______________________________

Question:	1	2	3	4	Total
Points:	40	30	10	20	100
Score:					

Question 1 .... *40 point*

Let A and B be two matrices such that $AB \neq BA$. Consider the functions:

$$\begin{aligned} E_0(t) &= e^{tA} &(1)\\ E_1(t) &= \int_0^t e^{(t-t_1)A} B e^{t_1 A} dt_1 = \int_0^t e^{(t-t_1)A} B E_0(t_1) dt_1 &(2) \end{aligned}$$

(a) (5 points) Show that

$$\frac{d}{dt} E_1(t) = AE_1(t) + BE_0(t)$$

(b) (5 points) Define recursively

$$\begin{aligned} E_{n+1}(t) &= \int_0^t e^{(t-t_1)A} B E_n(t_1) dt_1 = \\ &= \int_0^t dt_1 e^{(t-t_1)A} B \int_0^{t_1} dt_2 e^{(t_1-t_2)A} B \ldots \int_0^{t_{n-1}} dt_n e^{(t_{n-1}-t_n)A} B e^{t_n A} \end{aligned}$$

and show that

$$\frac{d}{dt} E_n(t) = AE_n(t) + BE_{n-1}(t)$$

(c) (10 points) Use the above realtion to show that

$$e^{(A+B)t} = \sum_{n=0}^{\infty} E_n(t)$$

(d) (10 points) Use the above to show that

$$\left.\frac{d}{dh} e^{A+hB}\right|_{h=0} = \int_0^1 e^{(1-s)A} B e^{sA} ds$$

(e) (10 points) Compute

$$\left.\frac{d^2}{dh^2} e^{A+hB}\right|_{h=0}$$

Question 2 . . . . *30 point*

Consider the system of equation

$$\begin{cases} \dot{x_1} = x_2 + x_1(x_1^2 + x_2^2)^\alpha \\ \dot{x_2} = -x_1 + x_2(x_1^2 + x_2^2)^\alpha \end{cases}$$

with $\alpha > -1$.

(a) (10 points) For which values of α does the system admit a unique solution for every initial condition?

(b) (10 points) For the values of α for which the solution is not unique, give an initial value and (at least) two solutions starting from it.

(c) (10 points) For which values of α does the system admit a global solution for every initial condition?

Question 3 . . . . *10 point*

Consider the differential equations:

$$\dot{X} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} X \tag{3}$$

and

$$\dot{Y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y \tag{4}$$

Write the conjugacy $Y = \phi(X)$ that conjugate eq.(3) and eq.(4).

Question 4 .... *20 point*

Consider the differential equation

$$\dot{X} = \begin{pmatrix} a & 1+b \\ 1-b & a \end{pmatrix} X$$

In the $a - b$ plane, describe the regions where the origin is a sink, a source, a saddle or a center. For each of these regions, give the general solution of the equation. Describe what happens on the boundary of these regions.