Possible Ideas for Personal Projects or Exercises

1) **The Dumped forced wave equation:** Consider a string of length 1 that vibrates in air and is externally forced. A possible description is through the following equation:

$$\begin{cases} \ddot{u}(x,t) = c^2 u"(x,t) - \gamma \dot{u}(x,t) + F(x,t) & 0 < x < 1 \\ u(0,t) = u(a,t) = 0 \\ u(x,0) = f(x) \\ \dot{u}(x,0) = f(x) \end{cases}$$

where $F(x,t)$ is periodic of period ω. Find the solution of this equation for general initial conditions. Discuss the long time behavior of the solutions as functions of γ and ω. You can start considering the simple case in which

$$F(x,t) = \sin(nx)\cos(\omega t)$$

We have seen this case in class. After you can generalize to

$$F(x,t) = f(nx)\cos(\omega t)$$

using the Fourier series for $f(x)$ and to

$$F(x,t) = \sin(nx) g(\omega t)$$

using the Fourier series for $g(t)$. Combining the above with the Fourier series in two dimension x and t) you will get the solution for the general case.

2) **Two Bars in convective contact:** Two bars of different materials are in contact through one extremity. The contact is convective. This means that the heat flowing from one bar to the other is proportional to the difference of temperature between the touching points. The two free extremities are kept at constant temperature. The equation for the system is

$$\begin{cases} \dot{u}(x,t) = k_1 u''(x,t) & 0 < x < 1 \\ \dot{u}(x,t) = k_2 u''(x,t) & 1 < x < 2 \\ u(0,t) = T_0 \\ u(2,t) = T_1 \\ k_1 u'(1^-,t) = -c\big(u(1^-,t) - u(1^+,t)\big) \\ k_2 u'(1^+,t) = c\big(u(1^-,t) - u(1^+,t)\big) \\ u(x,0) = f(x) \end{cases}$$

Find the general solution of this equation. For the steady state you will find an solution for $0 < x < 1$ and one for $1 < x < 2$ of the usual form. The coefficient will be fixed by the boundary conditions. In the same way separation of variable give you two equations the coefficient defining the eigenfunctions and the eigenvalues will be fixed by the boundary condition. Choosing reasonable value for the coefficient give an estimate of the first eigenvalue. solution

3) **Heat Equation in a Ring**: You have a circular plate of radius R with a circular hole of radius r in the center. The external boundary of the ring is at temperature T_R while the boundary of the hole is at temperature T_r. The equation governing the system is thus

$$\begin{cases} \dot{u}(x,y,t) = \Delta u(x,y,t) & r^2 < x^2 + y^2 < R^2 \\ u(x,y,t) = T_R & x^2 + y^2 = R^2 \\ u(x,y,t) = T_r & x^2 + y^2 = r^2 \\ u(x,y,0) = f(x,y) \end{cases}$$

Find the general solution of this equation. Observe that the eigenvalue for the radial part (whose eigenvector are solutions of a Bessel's equation) are given implicitly by the siltation of a linear system of equations. Use Matlab (or any other software of your election) to compute the first 5 eigenvalue.

4 **Numerical Methods**: implement with Matlab the solution of the heat equation in a bar with fixed boundary conditions. Assume that your bar is of length 1 and as temperature T_0 and T_1 at the boundaries. The equation you are looking at is then

$$\begin{cases} \dot{u}(x,t) = ku''(x,t) & 0 < x < 1 \\ u(0,t) = T_0 \\ u(1,t) = T_1 \\ u(x,0) = f(x) \end{cases}$$

Your code should be able to compute the solution for a general function $f(x)$. The projects should contain the description of a sample run. You can use all the standard packages of Matlab for diagonalizing matrices. After this you can generalize your program to convective boundary conditions, *i.e.* the new equation is

$$\begin{cases} \dot{u}(x,t) = ku''(x,t) & 0 < x < 1 \\ u'(0,t) = -c\big(T_0 - u(0,t)\big) \\ u'(1,t) = -c\big(T_1 - u(1,t)\big) \\ u(x,0) = f(x) \end{cases}$$

Observe that the only difference from the previous case is choosing the entries in the discrete Laplacian (second derivative) relative to the boundary terms. Finally assume that the bars in in convective contact with a medium at temperature $T(x)$. You may also assume that the convectivity $q(x)$ varies with x. The equation becomes

$$\begin{cases} \dot{u}(x,t) = ku''(x,t) + q(x)\big(T(x) - u(x,t)\big) \qquad 0 < x < 1 \\ u'(0,t) = -c\big(T_0 - u(0,t)\big) \\ u'(1,t) = -c\big(T_1 - u(1,t)\big) \\ u(x,0) = f(x) \end{cases}$$

This will require including a new term $T(x)q(x)$ to the matrix for the discrete second derivative. But will also force to solve numerically the steady state equation.

5 **Non-Uniform String**: A string of length 1 has a small piece of metal attached to a point. This means that the density of the system is higher where the piece of metal is present. The equation is thus

$$\begin{cases} \ddot{u}(x,t) = c(x)^2 u"(x,t) \qquad 0 < x < 1 \\ u(0,t) = u(a,t) = 0 \\ u(x,0) = f(x) \\ \dot{u}(x,0) = f(x) \end{cases}$$

where

$$c(x) = \begin{cases} 2 & \text{for } a - 0.05 < x < a - 0.05 \\ 1 & \text{otherwise} \end{cases}$$

For some $0.05 < a < 0.95$. Find the vibration frequencies of the string. Observe that the equation you obtain from separation of variable as to be solved independently in the three domain: $0 < x < a - 0.05$, $a - 0.05 < x < a + 0.05$ and $a + 0.05 < x < 1$. To find the eigenvalues and eigenfunctions you will have to impose continuity and differentiability at the points of discontinuity of $c(x)$: $a - 0.05$ and $a + 0.05$. This will lead you to a system of linear equation. Use Matlab to plot the solution as a function of a. Can you tell the position of the metal piece from the first frequency.