Fall 04 | Name: ______________________
Math 4581 | Test 1 | Bonetto

1) The two extremities of a rod are kept at constant teperatures T_0 and T_1 while along its length it is in convective contact with a media at a temperature that varies linearly between T_0 and T_1 form 0 to a. This mean that the temperature of the rod is governed by the equation:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{1}{k}\dfrac{\partial^2 u(x,t)}{\partial x^2} - h\left(u(x,t) - T(x)\right) \quad 0 \le x \le a \\ u(0,t) = T_0 \\ u(a,t) = T_1 \\ u(x,0) = \dfrac{T_1 + T_0}{2} \end{cases}$$

where

$$T(x) = T_0 + \frac{T_1 - T_0}{a} x$$

and the initial temperature is assumed constant.

a) Find the temperature of the rod $u(x,t)$ as a function of t, *i.e.* solve the above equation.

b) Compute

$$d(t)^2 = \int_0^a (u(x,t) - v(x))^2 dx$$

where $v(x)$ is the steady state solution. (**Hint**: use Parseval's identity.)

c) Call "realxation time" the time $\bar{t}$ such that $d(\bar{t}) = d(0)/2$. Can you find a upper bound for $\bar{t}$? How does the relaxation time depend on h? (**Hint**: use that $-\lambda_n^2 t \le -\lambda_1^2 t$ to estimate the exponentials in $d(t)$)

2) You hold the extremity of a semi-infinite string in your hand. The string is initially at rest. At time $t = 0$ you move it up at speed 1 for 0.25 seconds and then you move it down at speed 1 for 0.25 seconds . After time $t = 0.5$ second you hold it fixed at 0. This mean that the string is governed by the equation:

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = \dfrac{\partial^2 u(x,t)}{\partial x^2} \\ u(0,t) = h(t) \\ u(x,0) = 0 \\ \dfrac{\partial u(x,0)}{\partial t} = 0 \end{cases}$$

where

$$h(t) = \begin{cases} t & 0 < t < 0.25 \\ \\ 0.5 - t & 0.25 < t < 0.5 \\ \\ 0 & t > 0.5 \end{cases} .$$

We have assumed that the sound speed $c = 1$.

a) Use D'Alembert scheme to write the solution for every time $t > 0$.

b) Suppose now that the string has finite length $l = 2$. Write the solution for every time $t > 0$. (**Hint**: compute the state of the string at time $t = 0.5$ second and use it as initial condition to solve the wave equation with fixed extremities.)

c) Write and sketch $u(x,t)$ for $t = 3.25$ and $t = 4.25$ seconds. You may be able to do this without solving the point b).

3) A string of length 1 satisfy the wave equation, *i.e.* its displacement $u(x,t)$ satisfies:

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = c^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0 \\ \dfrac{\partial u(x,0)}{\partial t} = \epsilon g(x) \end{cases}$$

where the initail conditions $f(x)$ and $g(x)$ are given by:

$$g(x) = \begin{cases} \dfrac{1}{b} x & x < b \\ \dfrac{1}{1-b}(1-x) & x > b \end{cases}$$

a) The energy $E(t)$ of the string is given by:

$$E(t) = \int_0^1 \left(\partial_t u(x,t)^2 + \partial_x u(x,t)^2 \right) dx$$

Compute the energy of the string $E(t)$ for all $t > 0$.

b) Compute the solution using Fourier series.

c) Suppose now that the string is subject to an harmonic restoring force, *i.e.* it satisfies the equation

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = c^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} + \omega^2 u(x,t) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0 \\ \dfrac{\partial u(x,0)}{\partial t} = \epsilon g(x) \end{cases}$$

for a given ω. How will the previous solution change? (**Hint**: Use separation of variables and keep the ω term in the time equation. Differently you can write the solution $u(x,t)$ as sine Fourier series for every t and find an equation for the coefficients.)

d) **Bonus**: can you write an energy $E(t)$ for this new equation such that $\dot{E}(t) = 0$?

4) A rod of lenght a get a constant flux of heat Φ at one end and is in convective contact with a fluid at temperature T at the other end. Thus, the equation governing the temperature $u(x,t)$ inside the rod is:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{1}{k}\dfrac{\partial^2 u(x,t)}{\partial x^2} & 0 \leq x \leq a \\ \dfrac{\partial u(0,t)}{\partial x} = -\Phi & \\ \dfrac{\partial u(a,t)}{\partial x} = T - u(a,t) & \\ u(x,0) = T & \end{cases}$$

where we assumed that the convection constant $h = 1$ and that the initial temperature of the rod is constatnt and equal to T.

a) Write the equation for the steady state $v(x)$ and solve it.

b) Write the equation for the difference $w(x,t) = u(x,t) - v(x)$.

c) Use separation of variables to find the general solution for $w(x,t)$. You should find an equation for the eigenvalues λ_n. Do not try to sove it! Pay attention to the boudary condition.

d) Show that there are infinitely many eigenvalue λ_n and find an asymptotic value for them.

e) Write an expression for coefficients for the solution that satisfies the initial condition.

f) **Bonus**: write the solution of the problem.