1. Consider the periodic function

$$f(x) = x(|x| - 1) \qquad 1 \leq x \leq 1$$

(i) Does $f(x)$ has any symmetry?

(ii) Is it continuous? Is it sectionally continuous and sectionally smooth?

(iii) Compute $f'(x)$ and $f''(x)$. Are them continuous, sectionally continuous, sectionally smooth?

(iv) Compute the Fourier series of $f(x)$, $f'(x)$ and $f''(x)$.

(v) What can you say on the convergence of the Fourier series for $f(x)$, $f'(x)$ and $f''(x)$?

(vi) Let

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sin \frac{2n\pi}{a} x + \sum_{n=1}^{\infty} b_n \cos \frac{2n\pi}{a} x$$

Compute:

$$\sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2) \qquad \sum_{n=1}^{\infty} n^4(a_n^2 + b_n^2)$$

(vii) (Bonus) Let

$$f_N(x) = a_0 + \sum_{n=1}^{N} a_n \sin \frac{2n\pi}{a} x + \sum_{n=1}^{N} b_n \cos \frac{2n\pi}{a} x$$

Give an estimate of

$$\sup_x |f(x) - f_N(x)|$$

and

$$\int_{-1}^{1} |f(x) - f_N(x)|^2 dx$$

2. Let $f(x)$ be a continuous function of period a with Fourier series given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sin \frac{2n\pi}{a} x + \sum_{n=1}^{\infty} b_n \cos \frac{2n\pi}{a} x$$

(i) Find the Fourier series of

$$g(x) = \frac{f(x) + f(-x)}{2}$$

(ii) Find the Fourier series of

$$g(x) = \frac{f(x) - f(-x)}{2}$$

(iii) Find the Fourier series of

$$g(x) = f\left(2x + \frac{a}{2}\right)$$

3. The oscillation $u(t)$ of a pendulum are desribed by the equation

$$\ddot{u}(t) + \omega^2 u(t) = \cos(t)$$

Suppose the pendulum is initially at rest at its minimum, i.e. $u(0) = 0$. You want to hit it at time 0 in such a way that after 1 second the pendulum will be back at the minimum position, i.e. $u(1) = 0$. Which velocity $\dot{u}(0)$ should you give to the pendulum at time 0?