You can use your book and notes. No laptop or wireless devices allowed. Write clearly and try to make your arguments as linear and simple as possible. The complete solution of one exercise will be considered more that two half solutions.

Name: ______________________________

There are 5 questions for a total of 150pts. A total score of 100pts will grant you a full A for this midterm. You can chose which exercises to attempt. Remember that a full solution will be considered more that two half solutions.

Question:	1	2	3	4	5	Total
Points:	30	30	30	30	30	150
Score:						

1. (30 points) Given $A \in \mathbb{C}^{n\times n}$ define

$$w(A) = \sup_{\|x\|_2=1} |(x, Ax)|$$

and remember that

$$\|A\|_2 = \sup_x \frac{\|Ax\|_2}{\|x\|_2}$$

show that

(a) for every A we have $w(A) \le \|A\|_2$. (**Hint**: use the Cauchy-Schwartz inequality)

Solution:
For $\|x\|_2 = 1$ we have

$$|(x, Ax)| \le \|x\|_2\|Ax\|_2 = \frac{\|Ax\|_2}{\|x\|_2} \le \|A\|_2$$

(b) $w(A) = \|A\|_2$ if A is normal. (**Hint**: diagonalize A.)

Solution: If A is normal then $A = UDU^*$ and $A^* = UD^*U^*$ where D is diagonal. Thus $AA^* = UDD^*U^*$.
Let d_1 be the eigenvalue of A with largest modulus. It follows that

$$w(A) = \sup_{\|x\|_2=1} |(x, Dx)| = |d_1|$$

while

$$\|A\|_2 = \sup_x \frac{\|Ax\|_2}{\|x\|_2} = \sup_x \frac{\sqrt{(x, A^*Ax)}}{\|x\|_2} = \sup_x \frac{\sqrt{(x, D^*Dx)}}{\|x\|_2} = |d_1|$$

2. (30 points) Let $A \in \mathbb{C}^{2\times 2}$ be a Hermitian matrix such that $\operatorname{Tr} A > 0$ and $\operatorname{Det} A > 0$. Show that A admits a unique factorization $A = LL^*$, where L is lower triangular with positive (real) diagonal elements. (**Hint**: write down the equation for the component of L. What do the trace and the determinant tell you on the element of A?)

Solution: From the definition we have

$$\begin{aligned} l_{1,1}l^*_{1,1} &= a_{11} \\ l_{1,1}l^*_{2,1} &= a_{1,2} \\ l_{2,1}l^*_{2,1} + l_{2,2}l^*_{2,2} &= a_22 \end{aligned}$$

Since A is Hermitian we have $a_{1,1} = a^*_{1,1}$, $a_{2,2} = a^*_{2,2}$, and $a_{1,2} = a^*_{2,1}$.

Observe now that $\operatorname{Det} A = a_{1,1}a_{2,2} - a_{1,2}a^*_{1,2} > 0$. Since $a_{1,2}a^*_{1,2} > 0$ this implies $a_{1,1}a_{2,2} > 0$. Moreover since $\operatorname{Tr} A = a_{1,1} + a_{2,2} > 0$ we must have $a_{1,1} > 0$ and $a_{2,2} > 0$.

Thus from the first equation we get $l_{1,1} = \sqrt{a_{1,1}}$. The second gives

$$l_{2,1} = \frac{a^*_{1,2}}{l_{1,1}} = \frac{a^*_{1,2}}{\sqrt{a_{1,1}}}.$$

Finally the third read

$$|l_{2,2}|^2 = a_{2,2} - \frac{|a_{1,2}|^2}{a_{1,1}} > 0$$

3. (30 points) Suppose that $M \in \mathbb{C}^{n\times n}$ is in the block form

$$M = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$$

where $A \in \mathbb{C}^{n_1\times n_1}$ and $B \in \mathbb{C}^{n_2\times n_2}$, $n_1 + n_2 = n$.

Show that λ is an eigenvalue of M if and only if it is an eigenvalue of either A or B, or of both A and B. (**Hint**: look at the case $n = 2$ and $n_1 = n_2 = 1$ and then generalize.)

Solution: Suppose that λ is an eigenvalue of B with eigenvector v. Then the vector $x = (0, v^T)^T$ is an eigenvector of M with eigenvalue λ.

Suppose now that μ is an eigenvalue of A with eigenvector w and assume that μ is not an eigenvalue of B. Consider the vector $y = (w^T, u^T)^T$. We get

$$(M - \mu\mathrm{Id})\, y = \begin{pmatrix} (A - \mu\mathrm{Id})w \\ Cw + (B - \mu\mathrm{Id})u \end{pmatrix}.$$

But $(A-\mu\mathrm{Id})w = 0$ and, since $(B-\mu\mathrm{Id})$ is invertible, we can chose $u = (B-\mu\mathrm{Id})^{-1}Cw$ so that $(M - \mu\mathrm{Id})\, y = 0$.

Vice versa, if λ is an eigenvalue of M with eigenvector $x = (v^T, w^T)^T$ we must have $Av = \lambda v$. Thus either $v = 0$ or λ is an eigenvalue of A. If $v = 0$, we get $bw = \lambda w$ so that λ is an eigenvalue of B.

4. (30 points) Let $A \in \mathbb{C}^{n\times n}$ and let $p(z)$ be a polynomial. Show that μ is an eigenvalue of $B = p(A)$ if and only if $\mu = p(\lambda)$ where λ is an eigenvlue of A. (**Hint**: Assume first that A is upper triangular.)

Solution: We know that we can find $B \in \mathbb{C}^{n\times n}$ such that $A = BTB^{-1}$ is upper triangular with the eigenvalue of A on the diagonal. Observe that $p(A) = Bp(T)B^{-1}$. Clearly $p(T)$ is upper triangular and $p(T)_{i,i} = p(T_{i,i})$.

5. (30 points) Let $A, B \in \mathbb{C}^{n \times n}$, and assume that at least one of A and B have distinct eigenvalues. Then, if A and B commute they are simultaneously diagonalizable.

Solution: Assume that A has all distinct eigenvalue. Let λ_i be the eigenvalue of A with eigenvector v_i. Since λ_i has characteristic 1 we have that if $Aw = \lambda_i w$ then $w = \mu v_i$.

From the commutation we get

$$ABv_i = BAv_i = \lambda_i Bv_i$$

so that it follows that $Bv_i = \mu v_i$ for some μ and v_i is an eigenvector of B.