1.5. Linear equation with constant coefficients.

In this section, we present the basic theory of a linear system of differential equations with constant coefficients. The interest in such systems arises through the linear variational equation near a constant solution of a nonlinear equation $\dot{x} = f(x)$.

Let A be a $d \times d$ real or complex matrix and consider the equation

$$\dot{x} = Ax\,, \tag{5.1}$$

where $x \in \mathbb{R}^d$ or $\mathbb{C}^d$. From Theorem 3.1, the initial value problem for (5.1) has a unique solution which depends continuously upon the initial data. Let $\varphi^t(\xi)$ be the solution of (5.1) which satisfies $\varphi^0(\xi) = \xi$. From Exercise 4.2, the function $\varphi^t(\xi)$ is defined for all $t \in \mathbb{R}$. Suppose that $\varphi^t(\xi)$ and $\varphi^t(\zeta)$ are solutions of (5.1) and a, b are scalars. From the fact that the vector field in (5.1) is linear in x and there is a unique solution of the initial value problem for (5.1), we know that $\varphi^t(a\xi + b\zeta) = a\varphi^t(\xi) + b\varphi^t(\zeta)$. In particular, the mapping φ^t is a bounded linear operator on $\mathbb{R}^d$ or $\mathbb{C}^d$. Therefore, it is represented by a $d \times d$ matrix, which we designate by e^{At}; that is, for each $t \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$ or $\mathbb{C}^d$, we have

$$\varphi^t(\xi) = e^{At}\xi\,. \tag{5.2}$$

It is clear that each column of the matrix e^{At} is a solution of (5.1). In this sense, we say that e^{At} is a *matrix solution* of (5.1).

The matrix e^{At} satisfies the following properties:

$$\begin{aligned}
(i) e^{A0} &= I\,, \text{ the identity} \\
(ii) e^{A(t+s)} &= e^{At}e^{As} \\
(iii) (e^{At})^{-1} &= e^{-At} \\
(iv) \frac{d}{dt}e^{At} &= Ae^{At} = e^{At}A \\
(v) e^{At} &= I + At + \cdots + \frac{1}{n!}A^n t^n + \cdots\,.
\end{aligned}$$

Properties (i) was proved above. Property (ii) is a consequence of the uniqueness of solutions since, for any $\xi \in \mathbb{R}^d$, the functions $e^{A(t+s)}\xi$ and $e^{At}e^{As}\xi$ are solutions of the equation with the same initial values. Property (iii) is a consequence of (ii) by setting $s = -t$. Property (iv) follows from the uniqueness of the solution of the initial value problem and the observation that both Ae^{At} and $e^{At}A$ are matrix solutions of (5.1).

Before proving (v), we must make a few remarks about the meaning of a matrix power series. If $f(z)$ is a function of the complex variable z in a neighborhood of 0, the power series $f(A)$ is defined as the formal power series obtained by substituting for each term in the power series of $f(z)$ the matrix A for z. We say that $f(A)$

converges in a neighborhood of 0 if each element of the matrix $f(A)$ converges in a neighborhood of 0.

We could have taken the power series expansion in (v) as the definition of e^{At} and then show that it has all of the properties stated and that the matrix gave the solutions of (5.1) through the expression (5.2). On the other hand, we did not proceed this way. We used the differential equation (5.1) to define a linear operator which we called formally e^{At}. We use the method of successive approximations (*Picard iteration*) defined after the proof of Theorem 3.1 to show that the power series expansion of e^{At} is given by (v). More precisely, if we let

$$P^{(0)} = I,$$
$$P^{(k+1)}(t) = I + \int_0^t AP^{(k)}(s)\,ds, \quad k = 0,\, 1\, \dots\,,$$

then we see that the expression for $P^{(k)}$ is given by the formula

$$P^{(k)}(t) = I + At + \cdots + \frac{1}{n!}A^k t^k$$

which is the first k terms in (v). It remains to show that each element of the sequence of matrices $\{\, P^{(k)}(t),\, k = 0,\, 1,\, \dots,\, \}$ converges. There is a $\beta > 0$ such that, for any $d \times d$ matrix $B = \{\, b_{ij} \,\}$, we have $|b_{ij}| \le \beta^{-1}|B|$, where $|B|$ denotes the norm of B. If $P^{(k)}(t) = \{\, p_{ij}^{(k)}(t) \,\}$ and $|A| = \alpha$, then

$$\beta|p_{ij}^{(k)}(t)| \le 1 + \alpha t + \cdots + \frac{1}{k!}\alpha^k t^k \le e^{\alpha t}\,,$$

$$\beta|p_{ij}^{(k+m)}(t) - p_{ij}^{(k)}(t)| \le \Sigma_k^{k+m}\frac{1}{n!}\alpha^n t^n\,.$$

This show that each element of the sequence of matrices $\{\, P^{(k)}(t),\, k = 0,\, 1,\, \dots,\, \}$ is a Cauchy sequence. This proves (v).

Exercise 5.1. Prove the following facts:
(i) $Be^{At} = e^{At}B$ for all t if and only if $BA = AB$,
(ii) $e^{(A+B)t} = e^{At}e^{Bt}$ for all t if and only if $BA = AB$.

In the case where $d = 1$, we have shown above that the scalar differential equation $\dot{x} = ax$ defines in a unique way the exponential function e^{at}. For the matrix case, we have shown that (5.1) defines in a unique way an exponential matrix function. It remains to understand the more detailed structure of this matrix function e^{At}.

Exercise 5.2. Obtain e^{At} for the following differential equations using only the knowledge that we have presented above:
(i) $\dot{x} = ix,\ i = (-1)^{1/2}\,,$

(ii)$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1\,.$

There is an intimate connection between the eigenvalues and eigenvectors of A and the solutions of (5.1). We recall that λ is an *eigenvalue* of A if and only if det $[A-\lambda I] = 0$ and an *eigenvector* corresponding to λ is a nonzero vector ξ such that $[A-\lambda I]\xi = 0$. The function $x(t) = e^{\lambda t}\xi$ satisfies $\dot{x}(t) = e^{\lambda t}\lambda\xi = Ae^{\lambda t}\xi = Ax(t)$ and is thus a solution of (5.1). Conversely, if there is a complex number λ and a nonzero vector ξ such that $e^{\lambda t}\xi$ is a solution of (5.1), then λ is an eigenvalue and ξ is an eigenvector corresponding to λ. We also have the following result.

Theorem 5.1. *If $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of A and there is a basis $\xi_1, \ldots, \xi_d$ of eigenvectors, then*

$$e^{At} = [\xi_1 e^{\lambda_1 t}, \ldots, \xi_d e^{\lambda_d t}][\xi_1, \ldots, \xi_d]^{-1}\,. \tag{5.3}$$

Proof. Each column of the right hand side of (5.3) satisfies (5.1) and the matrix when evaluated at $t = 0$ is the identity. The same property holds for e^{At}. Therefore, by uniqueness of the solutions of the initial value problem, we have the assertion in the theorem.

If there is not a basis of eigenvectors, then the structure of the matrix e^{At} is more complicated. We now give a constructive method for determining e^{At}.

For any matrix B, let $N(B)$ designate the *null space* of B; that is, the set of vectors ξ such that $B\xi = 0$. If λ is an eigenvalue of A, let $r(\lambda)$ be the least integer k such that $N((A-\lambda I)^{k+1}) = N((A-\lambda I)^k)$. The *generalized eigenspace* $M_\lambda(A)$ of A corresponding to the eigenvalue λ is defined to be $N((A-\lambda I)^{r(\lambda)})$ and has dimension equal to the multiplicity of the eigenvalue λ. The subspace $M_\lambda(A)$ is invariant under A and thus, if $\xi \in M_\lambda(A)$, then $e^{At}\xi \in M_\lambda(A)$ for all t. Therefore, from (v), we have the polynomial representation of e^{At} on $M_\lambda(A)$ given by

$$e^{At}\xi = e^{\lambda t}e^{(A-\lambda I)t}\xi = e^{\lambda t}\Sigma_{k=0}^{r(\lambda)-1}(A-\lambda I)^k\frac{t^k}{k!}\xi, \quad \xi \in M_\lambda(A)\,. \tag{5.4}$$

Relation (5.4) shows that the solutions in $M_\lambda(A)$ are represented as polynomials in t times $e^{\lambda t}$. However, the degree of the polynomial may be less that $r(\lambda)-1$. As we determine a basis for $M_\lambda(A)$, we will recognize the number of blocks and the exact degree.

Relation (5.4) gives the solution on $M_\lambda(A)$. To continue, we need the following result. If $\lambda_1, \ldots, \lambda_q$ are the distinct eigenvalues of A, then

$$\mathbb{C}^d = \oplus_{j=1}^q N(A-\lambda_j I)^{r(\lambda_j)}\,,$$

where the symbol $\oplus$ means to take the span of the basis vectors of all of these subspaces. Using (5.4), we see that e^{At} is the sum of exponential functions with coefficients which are polynomials in t.

Example 5.1. We consider the equation corresponding to the linear oscillator:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 \,.$$

The eigenvalues of the corresponding matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are $\pm i$ and they are simple with eigenspaces spanned by the vectors $v_1 = \text{col}\ [i,\, 1]$, $v_2 = \text{col}\ [-i,\, 1]$, respectively. Therefore, $e^{At}v_1 = e^{it}v_1$, $e^{At}v_2 = e^{-it}v_2$. If $\xi = \text{col}\ (\xi_1,\, \xi_2) \in \mathbb{R}^2$, then it is represented uniquely in this basis of $\mathbb{C}^2$ as $\xi = \frac{1}{2i}(v_1 - v_2)\xi_1 + \frac{1}{2}(v_1 + v_2)\xi_2$ and a simple computation shows that

$$e^{At}\xi = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

Exercise 5.3. Find e^{At} for each of the following matrices A:

$$(a) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$(c) \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$(d) \begin{bmatrix} 0 & 1 & 0 \\ 4 & 3 & -4 \\ 1 & 2 & -1 \end{bmatrix}$$

It is possible to obtain very precise information about the structure of the matrix e^{At} from the *Jordan canonical form of A* (see Appendix). In fact, there exists a nonsingular $d \times d$ matrix P such that $P^{-1}AP$ is in Jordan canonical form:

$$P^{-1}AP = \ \text{diag}(A_1,\, A_2,\, \ldots,\, A_s)$$

where $A_j = \lambda_j I + N_j$ is a $d_j \times d_j$ matrix, N_j is the nilpotent matrix

$$N_j = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

and λ_j is an eigenvalue of A.

This implies that

$$P^{-1}e^{At}P = \operatorname{diag}(e^{A_1 t}, e^{A_2 t}, \ldots, e^{A_s t})$$

where $e^{A_j t} = e^{\lambda_j t}e^{N_j t}$. We easily compute the matrix $e^{N_j t}$ explicitly using (v) to obtain

$$e^{N_j t} = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \ldots & \frac{t^{d_j}}{d_j!} \\ 0 & 1 & t & \ldots & \frac{t^{d_j-1}}{(d_j-1)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & t \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

As a consequence, we obtain the solution $\varphi^t(\xi)$ of (5.1) as

$$\varphi^t(\xi) = e^{At}\xi = Pe^{P^{-1}APt}P^{-1}\xi\,.$$

Our next objective is to obtain some elementary results on the asymptotic behavior of the solutions of (5.1). The *spectrum* $\sigma(A)$ of a matrix A is the set of eigenvalues of A. We use the notation Re $\sigma(A) > 0$ (or < 0) to designate that Re $\lambda > 0$ (or < 0) for all $\lambda \in \sigma(A)$.

Lemma 5.1. *If Re $\sigma(A) < 0$, then there are positive constants k, α such that*

$$|e^{At}| \le ke^{-\alpha t},\ t \ge 0\,. \tag{5.5}$$

Proof. From the hypothesis, we know that there is an integer p such that the solutions of (5.1) are polynomials in t of degree $\le p$ multiplied by decaying exponentials. Since each column of e^{At} satisfies (5.1), the estimate (5.4) follows.

If A is a $d \times d$ matrix, we let A^* be the conjugate transpose of A. A $d \times d$ matrix B is *positive definite* if the quadratic form x^*Bx is positive definite on d-vectors x; that is, this quadratic form is positive for each $x \neq 0$. Notice that $x^*Bx = x^*(\frac{B+B^*}{2})x$ for all x and that $B + B^* = (B + B^*)^*$ is self adjoint. Therefore, without loss of generality, if B is positive definite, we may suppose that $B^* = B$ and we will write $B > 0$.

Suppsose that $B > 0$ and let $V(x) = x^*Bx$. Then there are positive constants m, M (given by the smallest and largest eigenvalue of B) such that

$$mx^*x \le V(x) \le Mx^*x \text{ for all } x.$$

Therefore, V is equivalent to a norm on $\mathbb{R}^d$.

Theorem 5.2. *(A Lyapunov theorem for linear systems) Suppose that $B > 0$ and let $V(x) = x^*Bx$. If there exists a positive definite matrix C such that the derivative $\dot{V}(x(t))$ along the solutions of (5.1) is*

$$\dot{V}(x) = x^*(A^*B + BA)x = -x^*Cx,$$

then (5.5) is satisfied; that is, $\operatorname{Re}\sigma(A) < 0$.

Exercise 5.4. Prove Theorem 5.2 and interpret this result in terms of the properties of the vector field on a level set of V; that is, for a fixed $c > 0$, the set $V^{-1}(c) = \{\, x : V(x) = c \,\}$. Use differential inequalities to obtain a bound on $x(t)$ for $t \geq 0$ in terms of the eigenvalues of B, C.

It is an important fact that the converse of Theorem 5.2 is true.

Theorem 5.3. *(A converse theorem of Lyapunov for linear systems) If* $\operatorname{Re}\sigma(A) < 0$, *then, for any $C > 0$, there is a $B > 0$ such that $A^*B + BA = -C$; that is, the function $V(x) = x^*Bx$ satisfies the conditions of Theorem 5.2.*

Exercise 5.5. Prove Theorem 5.3. *Hint.* Let $B = \int_0^\infty e^{A^*t}Ce^{At}\,dt$.

Example 5.2. Consider the damped linear oscillator

$$\ddot{x}_1 + \dot{x}_1 + x_1 = 0$$

or, equivalently, the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2.$$

The eigenvalues of the matrix of this system have negative real parts and so the solutions approach zero exponentially. Let us try to use Exercise 5.4 to obtain the same result. The energy (kinetic plus potential) for this system is $E(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$. We have

$$\dot{E}(x_1, x_2) = -x_2^2 \leq 0;$$

that is, the energy is strictly decreasing except when $x_2 = 0$. Without some other observations (which we mention in a later section), we cannot conclude that the solutions approach zero, much less exponentially. The energy does not satisfy the conditions of the function V in Exercise 5.4. On the other hand, we know there must be a positive definite quadratic form whose derivative along the solutions is negative definite. How do we find it? What are the level sets? Note that they cannot be symmetric with respect to the x_2-axis. Let us try

$$V(x_1, x_2) = \frac{1}{2}x_1^2 + \beta x_1 x_2 + \frac{1}{2}x_2^2$$

for some β. If $|\beta| < 1$, this is a positive definite quadratic form and

$$\dot{V}(x_1, x_2) = -[\beta x_1^2 + \beta x_1 x_2 + (1-\beta) x_2^2].$$

For $0 < \beta < \frac{4}{5}$, the matrix for the quadratic form in the brackets is positive definite. Thus, $\dot{V}(x_1, x_2)$ is negative definite and we can use Exercise 5.4 to conclude that the solutions approach zero exponentially. Draw the picture of the level sets of V.

We say that the differential equation $\dot{x} = Ax$ is *hyperbolic* if Re $\lambda \neq 0$ for all $\lambda \in \sigma(A)$. In this case, by a nonsingular transformation of variables, we can assume that $A = \mathrm{diag}\,(A_-,\, A_+)$, where Re $\sigma(A_-) < 0$ and Re $\sigma(A_+) > 0$. From Lemma 5.1, there are positive constants k, α such that

$$\begin{aligned} &|e^{A_- t}| \le k e^{-\alpha t},\ t \ge 0\,, \\ &|e^{A_+ t}| \le k e^{\alpha t},\ t \le 0\,. \end{aligned} \tag{5.6}$$

The second inequality in (5.6) is obtained by replacing t by $-t$ in the equation $\dot{z} = A_+ z$. If we partition $x \in \mathbb{R}^d$ as $x = \mathrm{col}\ (y,\, z)$ with the vector y (resp. z) having the same dimension as A_- (resp. A_+), then the linear subspace $W^s = \{\, x : x = \mathrm{col}(y,\, 0)\}$ (resp. $W^u = \{\, x : x = \mathrm{col}(0,\, z)\}$) are invariant under e^{At} and $e^{At}\xi$ approaches zero as $t \to \infty$ (resp. $t \to -\infty$) if $\xi \in W^s$ (resp. $\xi \in W^u$). The set W^s (resp. W^u) is called the *stable manifold* (resp. *unstable manifold*) of 0. When both of the subspaces W^s and W^u have dimension ≥ 1 (or, equivalently, when there is at least one eigenvalue with positive real part and at least one with negative real part), we refer to 0 as a *saddlepoint*.

Exercise 5.6. Suppose that $d = 2$ and that the planar system (5.1) is hyperbolic with the eigenvalues of A distinct. Show that there is a change of coordinates such that the flow is one of the cases shown in Figure 5.1. The arrows designate the direction of motion with increasing time.

Figure5.1a. Saddle point. **Figure5.1b**. Stable focus. **Figure5.1c**. Stable node.

Figure5.1d. Unstable focus. **Figure5.1e.** Unstable node.

æ