1.7. Stability and attractors.

Consider the autonomous differential equation

$$\dot{x} = f(x)\,, \tag{7.1}$$

where $f \in C^r(\mathbb{R}^d, \mathbb{R}^d)$, $r \geq 1$.

For notation, for any $x \in \mathbb{R}^d$, $c \in \mathbb{R}$, we let $B(x, c) = \{\, \xi \in \mathbb{R}^d : |\xi - x| < c \,\}$. Suppose that x_0 is an equilibrium point of (7.1). We say that x_0 is *stable* if, for any $\epsilon > 0$, there is a $\delta > 0$ such that, if $\xi \in B(x_0, \delta)$, then $\varphi^t(\xi) \in B(x_0, \epsilon)$ for $t \geq 0$. We say that x_0 is *unstable* if it is not stable. We say that x_0 *attracts points locally* if there exists a constant $b > 0$ such that, if $\xi \in B(x_0, b)$, then $|\varphi^t(\xi) - x_0| \to 0$ as $t \to \infty$; that is, for any $\eta > 0$ and any $\xi \in B(x_0, b)$, there is a $t_0(\eta, \xi)$ with the property that $\varphi^t(\xi) \in B(x_0, \eta)$ for $t \geq t_0(\eta, \xi)$. We say that x_0 is a *local attractor* if there exists a constant $c > 0$ such that dist $(\varphi^t(B(x_0, c), x_0) \to 0$ as $t \to \infty$; that is, for any $\eta > 0$, there is a $t_0(\eta)$ with the property that, if $\xi \in B(x_0, c)$, then $\varphi^t(\xi) \in B(x_0, \eta)$ for $t \geq t_0$. If x_0 has the property that, for any bounded set $B \subset \mathbb{R}^d$, we have dist $(\varphi^t(B), x_0) \to 0$ as $t \to \infty$, then we say that x_0 is a *global attractor* of (7.1).

To force us to think some about these definitions, let us consider in some detail the stability properties of the origin for the flow depicted in Figure 7.1. We do not write the equation, but just assume that there is an equation for which this is the flow (it is possible to show that such an equation does exist).

Figure 7.1.

The origin attracts points locally since every solution approaches zero as $t \to \infty$. On the other hand, the origin is not a local attractor. In fact, there is an $\eta > 0$ such that, for any sequence $\tau_k \in (0, \infty)$, $\tau_k \to \infty$ as $k \to \infty$, there is a sequence of points $\xi_k \to 0$ as $k \to \infty$ such that $\varphi^{\tau_k}(\xi_k) \notin B(0, \eta)$ for any k. For this same reason, the origin is not stable.

Exercise 7.1. Give a detailed discussion of the difference between stability of an equilibrium point x_0 and the continuous dependence on initial data in a neighborhood of x_0. Give an example where there is continuous dependence and not stability.

For the linear autonomous equation considered in Section 1.5, there is a simple criterion for determining the stability properties of the origin.

Theorem 7.1. *For the linear system (5.1), $\dot{x} = Ax$, we have the following statements: (i) The origin is stable if and only if Re $\sigma(A) \le 0$ and the eigenvalues with zero real parts have simple elementary divisors; that is, each Jordan block has dimension one; (ii) The origin is a global attractor for (5.1) if an only if Re $\sigma(A) < 0$.*

Proof. If the origin is stable, then we must have Re $\sigma(A) \le 0$. Furthermore, if there is a Jordan block of A which has dimension ≥ 2 and corresponds to an eigenvalue of A on the imaginary axis, then (5.3) implies that there is a solution of (5.1) which is t times a periodic function, which implies that the origin is unstable. The proof of (i) in the other direction is just as simple.

If the origin is a global attractor for (5.1), then every solution of (5.1) approaches zero. From (5.3), this implies that Re $\sigma(A) < 0$. If Re $\sigma(A) < 0$, then Lemma 5.1 implies that the origin is a global attractor.

Lemma 7.1. *An equilibrium point x_0 of (7.1) is stable and attracts points locally if and only if it is a local attractor.*

We do not present the proof since a more general result will be given below.

In the literature, the concept *the equilibrium point x_0 is asymptotically stable* is equivalent to x_0 being stable and attracting points locally.

We need also a generalization of the above concepts to invariant sets. For notation, for any set $J \subset \mathbb{R}^d$, $c \in \mathbb{R}$, we let $B(J, c) = \{ \xi \in \mathbb{R}^d : \mathrm{dist}(\xi, J) < c \}$. Suppose that J is an invariant set of (7.1). We say that J is *stable* if, for any $\epsilon > 0$, there is a $\delta > 0$ such that, if $\xi \in B(J, \delta)$, then $\varphi^t(\xi) \in B(J, \epsilon)$ for $t \ge 0$. We say that J is *unstable* if it is not stable. We say that J *attracts points locally* if there exists a constant $b > 0$ such that, if $\xi \in B(J, b)$, then $\mathrm{dist}(\varphi^t(\xi), J) \to 0$ as $t \to \infty$. We say that J is a *local attractor* if there exists a constant $c > 0$ such that dist $(\varphi^t(B(J, c), J) \to 0$ as $t \to \infty$; that is, for any $\eta > 0$, there is a $t_0(\eta)$ with the property that, if $\xi \in B(J, c)$, then $\varphi^t(\xi) \in B(J, \eta)$ for $t \ge t_0$. If J is a compact invariant set with the property that, for any bounded set $B \subset \mathbb{R}^d$, we have dist $(\varphi^t(B), J) \to 0$ as $t \to \infty$, then we say that J is a *global attractor* of (7.1). From this definition, it is clear there can be only one global attractor.

Lemma 7.2. *An invariant set J is stable if and only if, for any neighborhood V of J, there is a neighborhood $V' \subset V$ of J such that $\varphi^t(V') \subset V'$ for $t \ge 0$.*

Proof. If J is stable and V is a neighborhood of J, then there is a neighborhood W of J such that $\varphi^t(W) \subset V$ for $t \ge 0$. If $V' = \gamma^+(W)$, then $\varphi^t(V') \subset V'$ for $t \ge 0$. The converse is clear.

Theorem 7.2. *If J is a compact invariant set of (7.1), then J is stable and attracts points locally if and only if it is a local attractor.*

Proof. If J is a local attractor, then there is a positive constant c such that, for any $\epsilon > 0$, there is a t_0 such that $\varphi^t(B(J, c)) \subset B(J, \epsilon)$ for $t \ge t_0$. From continuity with respect to initial data, we can choose $0 < \delta < c$ so that $\varphi^t(B(J, \delta)) \subset B(J, \epsilon)$ for $0 \le t \le t_0$. Therefore, J is stable. Obviously, J attracts points locally.

Conversely, if J is stable and attracts points locally, then there is a bounded open neighborhood W of J such that dist $(\varphi^t(\xi), J) \to 0$ as $t \to \infty$ for every $\xi \in W$. From Lemma 7.2, we may assume that $\varphi^t(W) \subset W$ for $t \geq 0$. Let $U \subset W$ be an arbitrary neighborhood of J and let H be an arbitrary compact set in W. We only need to show that there is a $T(H, U)$ such that $\varphi^t(H) \subset U$ for $t \geq T(H, U)$. Without loss in generality, we also can take $\varphi^t(U) \subset U$, $t \geq 0$, from Lemma 7.2. For any $\xi \in H$, there is a $t_0 = t_0(\xi, U)$ such that $\varphi^t(\xi) \in U$ for $t \geq t_0$. Since φ^t is continuous, there is a neighborhood O_ξ of ξ such that $\varphi^{t_0}(O_\xi) \subset U$. If we let $\{ O_{\xi_j}, j = 1, 2 \ldots, p\}$ be a finite cover of H, $T(H, U) = \max_j t_0(\xi_j, U)$ and $V(H, U) = \cup_j O_{\xi_j}$, then $\varphi^t(V(H, U)) \subset U$ for $t \geq T(H, U)$. If we choose $H = B(J, b) \subset W$ and, for any $\eta > 0$, if we choose $U = B(J, \eta)$, then we have shown that J is a local attractor.

Example 7.2. For the equation, $\dot{x} = x(1 - x)$, the equilibrium points are 0, 1, with 0 being unstable and the point 1 being stable and attracts points locally. Therefore, the point 1 is a local attractor. There is no global attractor on $\mathbb{R}$.

Example 7.3. For the equation, $\dot{x} = x(1 - x^2)$, the equilibrium points are $0, \pm 1$, with 0 being unstable and the points ± 1 being stable and attracts points locally. Therefore, each of the points ± 1 is a local attractor. The line segment $[-1, 1]$ is a global attractor.

Example 7.4. For the equation,

$$\dot{x}_1 = -x_2 + x_1(1 - r^2), \quad \dot{x}_2 = x_1 + x_2(1 - r^2)\,,$$

with $r^2 = x_1^2 + x_2^2$, the only equilibrium point is $(0, 0)$ and it is unstable. The periodic orbit $r^2 = 1$ is a local attractor. The global attractor is the closed disk $\overline{B((0, 0), 1)}$.

We present now a general result on the existence of a global attractor. To do so, we need another concept. We say that (7.1) is *point dissipative* if there exists a bounded set $B \subset \mathbb{R}^d$ such that, for any $\xi \in \mathbb{R}^d$, there is a $t_0(\xi) \geq 0$ such that $\varphi^t(\xi) \in B$ for $t \geq t_0(\xi)$. To say that (7.1) is point dissipative is the same as saying that there is a bounded set B that attracts points globally. The following statement is a fundamental result in the theory of dissipative systems.

Theorem 7.3. *If (7.1) is point dissipative, then there exists a global attractor for (7.1). Furthermore, there is an equilibrium point of (7.1).*

Proof. Let B be a bounded open set such that, for any $\xi \in \mathbb{R}^d$, there is a $t_0(\xi)$ such that $\varphi^t(\xi) \in B$ for $t \geq t_0(\xi)$. We show first that, for any compact set $H \subset \mathbb{R}^d$, we have $\gamma^+(H)$ bounded. As in the proof of Theorem 7.2, there is a finite cover $\{ O_{\xi_j}, j = 1, 2 \ldots, p\}$ of H with the property that $\varphi^t(O_{\xi_j}) \subset B$ for $t = t_0(\xi_j)$. Define $T(H) = \max_j t_0(\xi_j)$. If $K = \mathrm{Cl}\, B$ and $\tilde{K} = \cup_{0 \leq t \leq T(K)} \varphi^t(K)$, then $\varphi^t(B) \subset \tilde{K}$ for $t \geq 0$ and, thus, $\varphi^t(H) \subset \tilde{K}$ for $t \geq T(H)$. Therefore, $\gamma^+(H)$ is bounded and it will have the its ω-limit set contained in $\tilde{K}$. On the other hand, $\gamma^+(\tilde{K})$ is bounded and so $\omega(\tilde{K})$ exists. Since $\omega(H) \subset \omega(\tilde{K})$ for every compact set H, we know that $\omega(\tilde{K})$ is the global attractor of (7.1).

From the asymptotic fixed point theorem (Theorem A.1.6), for any $\tau > 0$, there is an x_τ such that $\varphi^\tau(x_\tau) = x_\tau$. The function $\varphi^t(x_\tau)$ is thus a τ-periodic solution of (7.1). This periodic solution must lie in the global attractor which is compact. The existence of an equilibrium point of (7.1) is obtained by using the argument used in the proof of Theorem 6.4. This completes the proof of the theorem.

Remark 7.1. It is possible but nontrivial to give an example of a differential equation in $\mathbb{R}^3$ for which all solutions are bounded and yet there is no equilibrium point.

For a scalar differential equation, it is easy to determine if it is point dissipative. In fact, if f is a scalar vector field, then a necessary and sufficient condition that (7.1) be point dissipative is that there is an $r_0 > 0$ such that $xf(x) < 0$ for $|x| > r_0$. The global attractor is the interval $I = [x_0, x_1]$, where x_0 (resp., x_1) is the smallest (resp., largest) zero of f which is larger that $-r_0$ (smaller that r_0). The pictorial situation is shown in Figure 7.2.

Figure 7.2. One dimensional dynamics.

This one dimensional example is a special case of a more general class of differential equations on $\mathbb{R}^d$ called gradient systems.

Example 7.5. (*Gradient systems*) If $F \in C^r(\mathbb{R}^d, \mathbb{R})$, $r \geq 1$, and ∇F denotes the gradient of F, then the equation

$$\dot{x} = -\nabla F(x) \tag{7.2}$$

is called a *gradient system.*

The set E of equilibrium points of (7.2) coincides with the set of extreme points of the function F. Furthermore, we have the following relation:

$$\frac{d}{dt} F(\varphi^t(\xi)) = -|\nabla F(\varphi^t(\xi))|^2 \leq 0 \tag{7.3}$$

for all $\xi \in \mathbb{R}^d$, where $|x|^2 = x \cdot x$. This implies that

$$F(\varphi^t(\xi)) \leq F(\xi) \text{ for all } t \geq 0\,. \tag{7.4}$$

If $\varphi^t(\xi)$ is defined and bounded for all $t \geq 0$, then we know that $\omega(\xi)$ is a nonempty, compact and invariant set. Since $F(\varphi^t(\xi))$, $t \geq 0$, is bounded below,

$F(\varphi^t(\xi)) \to$ a limit as $t \to \infty$. This implies that, for any $\zeta \in \omega(\xi)$, we have $F(\varphi^t(\zeta)) = F(\zeta)$ for all $t \in \mathbb{R}$. Therefore, $\frac{d}{dt}F(\varphi^t(\zeta)) = 0$ for all $t \in \mathbb{R}$. From (7.3), this implies that $\nabla F(\varphi^t(\zeta)) = 0$ for all $t \in \mathbb{R}$ and $\omega(\xi) \subset E$.

If we assume that the function F satisfies the additional property that

$$F(x) \to \infty \text{ as } |x| \to \infty\,, \tag{7.5}$$

then (7.4) implies that $\varphi^t(\xi)$ is defined and bounded for $t \geq 0$. As a consequence, we have proved the following result.

Theorem 7.4. *If $\gamma^+(\xi)$ is a bounded orbit of the gradient system (7.2), then $\omega(\xi)$ belongs to the set of equilibrium points. If, in addition, each equilibrium point of (7.2) is isolated, then $\omega(\xi)$ is a single point. Finally, if (7.5) is satisfied, then each $\gamma^+(\xi)$ is bounded.*

Proof. We only need to prove that $\omega(\xi)$ is a single point if each equilibrium point is isolated. This follows from the fact that the ω-limit set is connected.

Remark 7.2. In most applications of gradient systems, we have $\omega(\xi)$ is a single point. On the other hand, it is possible to give an example in the plane for which this is not true. It is instructive to try to construct such an example.

The structure of the flow on the global attractor for a gradient system can be described in terms of the unstable set of equilibrium points. If ξ is an equilibrium point of (7.2), we define the unstable set of ξ as

$$W^u(\xi) = \{\, x : \varphi^t(x) \text{ exists on } (-\infty,\, 0],\, \varphi^t(x) \to \xi \text{ as } t \to -\infty \,\}\,.$$

In the same way, we define the unstable set of the set E of equilibrium points as

$$W^u(E) = \{\, x : \varphi^t(x) \text{ exists on } (-\infty,\, 0],\, \varphi^t(x) \to E \text{ as } t \to -\infty \,\}\,.$$

Theorem 7.5. *Suppose that (7.2) has a global attractor $\mathcal{A}$. Then $\mathcal{A} = W^u(E)$. If, in addition, the equilibrium points are isolated, then*

$$\mathcal{A} = \cup_{\xi \in E} W^u(\xi).$$

Proof. If $x \in \mathcal{A}$, then $\varphi^t(x)$ exists on $(-\infty,\, 0]$ and is bounded. Therefore, to prove the theorem, we only need to show that $\alpha(x) \in E$. If $y \in \alpha(x)$, then there exists a sequence $t_n \to -\infty$ as $n \to \infty$ so that $\varphi^{t_n}(x) \to y$ as $t \to \infty$. Choose the t_n so that $t_{n-1} - t_n \geq 1$ for all n. Then, for any $t \in (0,\, 1)$, we have from (7.4) that $F(\varphi^{t_{n-1}}(x)) \leq F(\varphi^{t_n + t}(x)) \leq F(\varphi^{t_n}(x))$ for all n and thus $F(\varphi^{t_n+t}(x)) \to F(y)$ as $n \to \infty$. Since $F(\varphi^{t_n}(x))$ also converges to $F(\varphi^t(y))$ as $n \to \infty$, it follows that $F(\varphi^t(y)) = F(y)$ for all $t \in [0,\, 1)$ and therefore for all $t \in \mathbb{R}$. Relation (7.4) implies that $y \in E$ and $\mathcal{A} = W^u(E)$. If E consists only of isolated points, then $W^u(E) = \cup_{\xi \in E} W^u(\xi)$ and the theorem is proved.

Exercise 7.2. Show that every one dimensional autonomous differential equation is a gradient system.

Exercise 7.3. Prove the following result: *If the set of equilibrium points of the gradient system (7.2) is bounded and (7.5) is satisfied, show that there is a global attractor for (7.2).*

Exercise 7.4. Give an example of a gradient system for which (7.5) is satisfied and yet there is no global attractor.

Exercise 7.5. Show that the following equations are gradient systems and prove that there is a global attractor:

$$
\begin{aligned}
&(a)\dot{x}_1 = 2(x_2 - x_1) + x_1(1 - x_1^2), \quad \dot{x}_2 = -2(x_2 - x_1) + x_2(1 - x_2^2),\\
&(b)\dot{x}_1 = -x_1^3 - bx_1x_2^2 + x_1, \quad \dot{x}_2 = -x_2^3 - bx_1^2x_2 + x_2\,.
\end{aligned}
$$

Exercise 7.6. Suppose that $\mathcal{A}$ is a compact invariant set that is a local attractor and, for any $\xi \in \mathbb{R}^d$, we have $\omega(\xi) \subset \mathcal{A}$. Prove that $\mathcal{A}$ is a global attractor.

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