1.8. Liapunov functions.

On several occassions, we have gained information about the behavior of the solutions of a differential equation by considering the rate of change of a scalar function along the solutions of the equation. In particular, in Exercises 4.3 and 4.4, we used a positive definite quadratic form to determine the stability properties of the origin. For linear systems, we have seen that the origin being a local attractor is equivalent to the existence of a positive quadratic form along which the derivative along solutions is negative definite. In the study of gradient systems, we showed that the scalar function that defined the system could be used to show that the system was point dissipative and therefore had a global attractor. There is a general theory based upon these ideas - the theory of Lyapunov functions - which we now describe.

We concentrate at first on stability of an equilibrium point which we may assume to be the origin.

Definition 8.1. Let U be a bounded open set of $\mathbb{R}^d$ containing the origin. A function $V \in C^k(U, \mathbb{R})$. $k \geq 0$ is said to be *positive definite* on U if $V(0) = 0$, $V(x) > 0$ for $x \neq 0$ in U. The function V is said to be *negative definite* if $-V$ is positive definite.

If $f(x)$ is a given d-dimensional vector field and $V \in C^1(U, \mathbb{R})$ is a given function, we define

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x)\,;$$

that is, $\dot{V}(x(t))$ is the derivative of the function V along the solutions $x(t) \in U$ of the differential equation $\dot{x} = f(x)$.

Theorem 8.1. *(Stability from Lyapunov functions) Suppose that $x = 0$ is an equilibrium point of the d-dimensional system $\dot{x} = f(x)$ and suppose that V is a positive definite function in a neighborhood U of 0. Then $\dot{V}(x) \leq 0$ in U implies that 0 is a stable equilibrium point and $\dot{V}$ negative definite in U implies that 0 is a local attractor.*

Proof. Let $\epsilon > 0$ be sufficiently small so that U contains the ball $B(0, \epsilon) \equiv \{x : |x| \leq \epsilon\}$ of center zero and radius ϵ. Since $\partial B(0, \epsilon)$ is closed and bounded, the minimum value m of V on $\partial B(0, \epsilon)$ exists and is positive. Since V is continuous and $V(0) = 0$, we can choose $\delta > 0$ such that $V(x) < m$ for $x \in B(0, \delta)$. If $|x_0| \leq \delta$, then the solution $x(t)$ through x_0 satisfies $V(x(t)) \leq V(x_0) < m$ for all $t \geq 0$. Therefore, $|x(t)| < \epsilon$ for $t \geq 0$ and the origin is stable.

Now let us suppose that $-\dot{V}$ is positive definite in U and choose a fixed $\epsilon > 0$ and a corresponding δ as in the proof of stability of the origin. For any $\eta > 0$, $0 < \eta < \delta$, there is a positive constant c such that $-\dot{V}(x) \geq c$ for $\eta \leq |x| \leq \epsilon$. As a consequence, if $x_0 \in B(0, \delta)$, the solution $x(t)$ through x_0 satisfies $V(x(t)) \leq V(x_0) - ct$ as long as $\eta \leq |x(t)| \leq \epsilon$. Since $|x(t)| \leq \epsilon$ for all $t \geq 0$ and $V(x) \geq 0$ for all $x \in U$, there must exist a $t_0 = t_0(\eta, \delta)$ such that $|x(t_0)| = \eta$. This shows that $x(t) \to 0$ as $t \to \infty$. Lemma 7.1 shows that the origin is a local attractor.

Exercise 8.1. For a given function $f \in C^k(\mathbb{R}^d, \mathbb{R}^d)$ and $V \in C^0(U, \mathbb{R})$, define

$$\dot{V}(x) = \limsup_{h \to 0^+} \frac{1}{h}[V(x + hf(x)) - V(x)].$$

With this definition of $\dot{V}$, give an appropriate generalization of Theorem 8.1.

Theorem 8.2. *(Stability from linearization) If $f \in C^r(\mathbb{R}^d, \mathbb{R}^d)$, $r \geq 1$, and there is an $x_0 \in \mathbb{R}^d$ such that $f(x_0) = 0$, and Re $\sigma(\partial f(x_0)/\partial x) < 0$, then the equilibrium solution x_0 of the equation $\dot{x} = f(x)$ is a local attractor.*

Proof. Without loss of generality, we can take $x_0 = 0$. If $f(x) = Ax + g(x)$, where $g(0) = 0$, $\partial g(0) = 0/\partial x$, then Re $\sigma(A) < 0$. From Section 1.5, there is a positive definite matrix B such that $A^*B + BA = -I$. If we let $V(x) = x^*Bx$, then there are positive constants $m < M$ such that $mx^*x \leq V(x) \leq Mx^*x$ for all x. Also, there is a positive constant b such that $\dot{V}(x)$ along the solutions of the equation $\dot{x} = Ax + g(x)$ is given by

$$\dot{V}(x) = -x^*x + 2x^*Bg(x) \leq -\frac{1}{2}x^*x \leq -\frac{m}{2}V(x)$$

for $|x| < b$. Therefore,

$$mx^*(t)x(t) \leq V(x(t)) \leq e^{-\frac{m}{2}t}V(x(0)) \leq Me^{-\frac{m}{2}t}x^*(0)x(0)$$

as long as $|x(t)| \leq b$. If we choose δ so that $M\delta^2 \leq b^2$, then $|x(0)| \leq \delta$ implies that the above inequality holds for all $t \geq 0$. This proves the corollary.

Exercise 8.2. Suppose that $V(x_1, x_2) = x_2^2 e^{-x_1}$ for $(x_1, x_2) \in \mathbb{R}^2$ and that, relative to some planar autonomous differential equation, we have $\dot{V}(x_1, x_2) = -x_2^2 V(x_1, x_2)$. Is it possible to say anything about the solutions of the original differential equation? If not, why?

It is possible to give a generalization of Theorem 8.1 which is valid for nonautonomous equations and time dependent Lyapunov functions. We leave the precise formulation for the reader.

If we exploit the autonomous character of our equation and especially the invariance of the ω-limit set, it is possible to give a very general result on the limit set of any bounded orbit

Theorem 8.3. *(The Invariance Principle) Suppose that $V \in C^1(\mathbb{R}^d, \mathbb{R})$, $U = \{x \in \mathbb{R}^d : V(x) < k\}$ and suppose that $\dot{V} \leq 0$ on U. Let*

$$S = \{x \in \bar{U} : \dot{V}(x) = 0\}$$

and let M be the largest set in S which is invariant for the equation $\dot{x} = f(x)$. Then every positive orbit in U which is bounded has its ω-limit set in M.

Corollary 8.1. *Suppose that the conditions of Theorem 8.3 are satisfied and, in addition, U is bounded and $\dot{V} < 0$ on ∂U. Then there is a compact invariant set $\mathcal{A}$ in U such that $\mathcal{A}$ is the local attractor for U; that is, $\mathcal{A}$ attracts U.*

Proof. The hypotheses imply that there is a closed bounded set W in U such that the ω-limit set of any orbit in U belongs to W. We now use the same type of argument as in the proof of Theorem 7.3.

Exercise 8.3. Prove Theorem 8.3.

Example 8.1. Consider the second order scalar equation

$$\ddot{x} + f(x)\dot{x} + h(x) = 0, \tag{8.1}$$

where $xh(x) > 0$, $x \neq 0$, $f(x) > 0$, $x \neq 0$, and $H(x) \equiv \int_0^x h(s)ds \to \infty$ as $|x| \to \infty$. The equivalent system is

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h(x_1) - f(x_1)x_2. \tag{8.2}$$

If $V(x_1, x_2) = x_2^2 + H(x_1)$, the total energy of the system, then $\dot{V} = -f(x_1)x_2^2$. For any positive number ρ, the function V is a Lyapunov function on the bounded set $U = \{(x_1, x_2) : V(x_1, x_2) < \rho\}$. Also, the set S where $\dot{V} = 0$ belongs to the union of the x_1-axis and the x_2-axis. From (8.2), this implies that the largest invariant set M in S is given by $M = \{(0, 0)\}$. Therefore, Theorem 8.3 implies that every solution of (8.2) approaches the origin. Since this implies that the system is point dissipative, Theorem 7.3 asserts that (8.2) has a global attractor $\mathcal{A}$. Since the global attractor is invariant and the only invariant set of (8.2) is the origin, it follows that the origin is a global attractor.

Exercise 8.4. Consider the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -2x_1 - 3x_1^2 - ax_2,$$

where $a > 0$. Use the energy function $V(x_1, x_2) = x_2^2/2 + x_1^2 + x_1^3$ to discuss the stable manifold of the origin.

Exercise 8.5. Consider the third order system

$$\dot{x} = y, \quad \dot{y} = z - ay, \quad \dot{z} = -cx - F(y),$$

where $F(0) = 0$, $a > 0$, $c > 0$, $aF(y)/y > c > 0$ for $y \neq 0$ and $\int_0^y [F(s) - cs/a]ds \to \infty$ as $|s| \to \infty$. If $F(y) = ky$, where $k > c/a$, verify that the characteristic roots of the linear system have negative real parts. Show that the origin is a global attractor even for the nonlinear function F satisfying the above properties.
Hint: Choose a Lyapunov function V as a quadratic form plus the term $\int_0^y F(s)ds$.

Exercise 8.6. Suppose that $f \in C^r(\mathbb{R}^d, \mathbb{R}^d)$, $r \geq 1$, and that there is a positive definite matrix Q such that $J'(x)Q + QJ(x)$ is negative definite for all $x \neq 0$, where $J'(x)$ is the Jacobian matrix of $f(x)$. If $f(0) = 0$, prove that the origin is a global attractor for the system $\dot{x} = f(x)$.
Hint: Prove and make use of the fact that $f(x) = \int_0^1 J(sx)ds$.

We state another important theorem concerning instability by using Lyapunov functions.

Theorem 8.4. *(Ĉetaev's Instability Theorem) Consider the differential equation $\dot{x} = f(x)$ and suppose that 0 is an equilibrium point. Let U be a bounded neighborhood of 0. If there exist an open set Ω and a function $V \in C^1(\bar{\Omega}, \mathbb{R})$ such that*
(i) $0 \in \partial\Omega$;
(ii) $V(x) = 0$ *for all* $x \in U \cap \partial\Omega$;
(iii) $V(x) > 0$ *and* $\dot{V}(x) > 0$ *for all* $x \in \Omega \cap \bar{U}$;

then the origin is an unstable equilibrium point.

Proof. If $x_0 \in (\Omega \cap U) \setminus \{0\}$ and the solution $x(t)$ through x_0 remains in U for all $t \geq 0$, then it must have an ω-limit set $\omega(x_0)$ in $\bar{U}$. As in the proof of Theorem 8.3, this implies that $\dot{V} = 0$ on $\omega(x_0)$. Our hypotheses imply that $\omega(x_0) = \{0\}$, which is a contradiction since $V > 0$ on $\omega(x_0)$.

Using Theorem 8.4, we can obtain the following result about instability of an equilibrium point based on the linear approximation.

Theorem 8.5. *(Instability from linearization) If $f \in C^r(\mathbb{R}^d, \mathbb{R}^d)$, $r \geq 1$, $f(x_0) = 0$, $A = \frac{\partial f(x_0)}{\partial x}$, Re $\sigma(A) \neq 0$ and there is a $\lambda \in \sigma(A)$ such that Re $\lambda > 0$, then x_0 is unstable.*

Proof. If we let $A = \frac{\partial f(x_0)}{\partial x}$, $y = x - x_0$ and $g(y) = f(y + x_0) - Ay$, then $g(y) = o(|y|)$ as $y \to 0$; that is, for any $\epsilon > 0$, there is a $\delta > 0$ such that $|g(y)| \leq \epsilon |y|$ for $|y| \leq \delta$. Also, y is a solution of the differential equation

$$\dot{y} = Ay + g(y)\,.$$

Without loss of generality, we may suppose that $A = \text{diag}\,(A_+,\, A_-)$ where the eigenvalues of A_+ (resp. A_-) have real parts > 0 (resp. < 0). If we let $y = \text{col}\,(u,\, v)$, $g = \text{col}\,(U,\, V)$, then the differential equation becomes

$$\begin{aligned} \dot{u} &= A_+ u + U(u,\, v) \\ \dot{v} &= A_- v + V(u,\, v)\,. \end{aligned} \tag{8.3}$$

Let B_+ (resp. B_-) be a positive definite matrix such that $-A_+^* B_+ - B_+ A_+ = -I$ (resp. $A_-^* B_- + B_- A_- = -I$). These matrices exist by Exercise 1.5.4. If we define $W(u,\, v) = u^* B_+ u - v^* B_- v$, then

$$\dot{W}(u, v) = u^* u + v^* v + 2u^* B_+ U(u, v) - 2v^* B_- V(u, v).$$

Since $g(y) = o(y)$ as $y \to 0$, for any $\epsilon \in (0, 1)$, there is a neighborhood $\Omega = \Omega_\epsilon$ of $(u, v) = (0, 0)$ such that

$$\dot{W}(u, v) \geq (1 - \epsilon)(|u|^2 + |v|^2) \text{ for } (u, v) \in \Omega.$$

The set $W^+ = \{(u, v) : W(u, v) > 0\}$ is an open set whose boundary ∂W^+ is a cone; that is, if $(u, v) \in \partial W^+$, then $(ku, kv) \in \partial W^+$ for all $k \in \mathbb{R}$. Thus, $(0, 0) \in \partial W^+$. If we choose $(u_0, v_0) \in \Omega \cap W^+$, then the solution $(u(t), v(t))$ of (8.3) through (u_0, v_0) must remain in $\Omega \cap W^+$ for all t as long as $(u(t), v(t))$ remains in Ω. Since $\dot{W}(u, v) > 0$ if $(u, v) \neq 0$, it follows that $(u(t), v(t)) \in \partial\Omega$ for some finite time t. Since the closure of the set W^+ contains the origin, this proves instability of the origin. The proof of the theorem is complete.

In a later section, we will see that the conclusion of instability remains true even if we do not assume the hyperbolicity of the linear approximation near the equilibrium point. However, the proof is more difficult.

The following result also is very important and gives a very geometric picture of the meaning of a local attractor.

Theorem 8.6. *(Converse Theorem of Lyapunov) If $\mathcal{A}$ is a local attractor for the d-dimensional system $\dot{x} = f(x)$, then there exists a neighborhood U of $\mathcal{A} \in \mathbb{R}^d$ and a Lipschitz continuous function V on U such that V is positive on $U \setminus \mathcal{A}$, $V = 0$ on $\mathcal{A}$, and $\dot{V}$ is negative on $U \setminus \mathcal{A}$. More precisely, there exist a positive constant r_0, a positive definite function $b(r)$, a positive function $c(r)$, $0 \leq r \leq r_0$, and a Lipschitz continuous function $V(x)$ defined for $x \in \mathbb{R}^d$, $|x| \leq r_0$, such that*

$$\begin{aligned} &(a)\ \operatorname{dist}(x, \mathcal{A}) \leq V(x) \leq b(\operatorname{dist}(x, \mathcal{A})), \\ &(b)\ \dot{V}(x) \leq -c(\operatorname{dist}(x, \mathcal{A}))V(x) \leq -\operatorname{dist}(x, \mathcal{A})c(\operatorname{dist}(x, \mathcal{A})), \end{aligned} \tag{8.4}$$

In particular, when the local attractor is the origin, we have the converse of Theorem 8.1, which is stated as

Theorem 8.7. *If 0 is a local attractor for the d-dimensional system $\dot{x} = f(x)$, then there exists a neighborhood U of $0 \in \mathbb{R}^d$ and a Lipschitz continuous function V on U such that V is positive definite and $\dot{V}$ is negative definite on U. More precisely, there exist a positive constant r_0, a positive definite function $b(r)$, a positive function $c(r)$, $0 \leq r \leq r_0$, and a Lipschitz continuous function $V(x)$ defined for $x \in \mathbb{R}^d$, $|x| \leq r_0$, such that*

$$\begin{aligned} &(a)\ |x| \leq V(x) \leq b(|x|), \\ &(b)\ \dot{V}(x) \leq -c(|x|)V(x) \leq -|x|c(|x|). \end{aligned} \tag{8.5}$$

We only indicate the proof for a special case; namely, the case in which the differential equation is linear, the local attractor is the origin and the solution $x(t, x_0)$ satisfies the inequality

$$|x(t, x_0) \leq Ke^{-\alpha t}|x_0|, \quad t \geq 0, \tag{8.6}$$

for all $x_0 \in \mathbb{R}^d$ and k, α are positive constants. The other situation is a little more complicated but uses similar ideas.

We define

$$V(x_0) = \sup_{\tau \geq 0} |x(\tau, x_0)| e^{\alpha\tau}.$$

From (8.6), it is clear that $V(x_0)$ is defined for all $x_0 \in \mathbb{R}^d$ and satisfies (8.5)(a). To show that V is Lipzchitzian, we observe that

$$\begin{aligned}
|V(x_0) - V(y_0)| &\leq \sup_{\tau \geq 0} [|x(\tau, x_0) - x(t, y_0)| e^{\alpha\tau} \\
&= \sup_{\tau \geq 0} |x(\tau, x_0 - y_0)| e^{\alpha\tau} \\
&= V(x_0 - y_0) \leq K |x_0 - y_0|.
\end{aligned}$$

The proof of (8.5)(c) procceds as follows:

$$\begin{aligned}
\dot{V}(x_0) &= \limsup_{h \to 0^+} \frac{1}{h} [\sup_{\tau \geq 0} |x(\tau + \tau, x_0)| e^{\alpha\tau} - \sup_{\tau \geq 0} |x(\tau, x_0)| e^{\alpha\tau}] \\
&= \limsup_{h \to 0^+} \frac{1}{h} [\sup_{\tau \geq h} |x(\tau + \tau, x_0)| e^{\alpha(\tau - h)} - \sup_{\tau \geq 0} |x(\tau, x_0)| e^{\alpha\tau}] \\
&\leq \limsup_{h \to 0^+} \frac{1}{h} [\sup_{\tau \geq 0} |x(\tau + \tau, x_0)| e^{\alpha(\tau - h)} - \sup_{\tau \geq 0} |x(\tau, x_0)| e^{\alpha\tau}] \\
&= \limsup_{h \to 0^+} \frac{1}{h} \sup_{\tau \geq 0} |x(\tau + \tau, x_0)| e^{\alpha\tau} (e^{-\alpha h} - 1) \\
&= -\alpha V(x_0).
\end{aligned}$$

This completes the proof for this special case.

We conclude this section with some remarks about how local attractors may change when the differential equation is subjected to perturbations. More precisely, consider the family of differential equations

$$\dot{x} = f(x, \epsilon), \tag{8.7$_\epsilon$}$$

where ϵ is a parameter in a Banach space E and $f(x, \epsilon)$ and $D_x f(x, \epsilon)$ are continuous in x, ϵ for $(x, \epsilon) \in \mathbb{R}^d \times E$.

Theorem 8.8. *If (8.7)$_0$ has a local attractor $\mathcal{A}_0$, then there is a positive number δ such that, for $0 \leq |\epsilon| < \delta$, the system (8.7)$_\epsilon$ has a local attractor $\mathcal{A}_\epsilon$ and the family of sets $\{\mathcal{A}_\epsilon,\ 0 \leq |\epsilon| < \delta\}$ is upper semicontinuous at $\epsilon = 0$; that is,*

$$\lim_{\epsilon \to 0} \operatorname{dist}(\mathcal{A}_\epsilon, \mathcal{A}_0) = 0.$$

Proof. We will make use of Theorem 8.6 even though a simple proof can be given directly (see Exercise 8.7). From Theorem 8.6, there is a neighborhood U of $\mathcal{A}_0 \in \mathbb{R}^d$ and a function $V \in C^1(U, \mathbb{R})$ such that V is positive on $U \setminus \mathcal{A}_0$, $V = 0$ on $\mathcal{A}_0$, and $\dot{V}$ is negative on $U \setminus \mathcal{A}_0$. If we let $V^{-1}(c) = \{x \in \mathbb{R}^d : \text{dist}\,(x, \mathcal{A}_0) < c\}$, then we can choose c and $\gamma > 0$ so that $V^{-1}(c)$ is a bounded set and $\dot{V} \leq -\gamma$ on $\partial V^{-1}(c)$. From the hypotheses of continuity of $f(x, \epsilon)$, there is a $\delta_0 > 0$ such that, for $0 \leq |\epsilon| < \delta_0$, we have $\dot{V}_{(8.7)_\epsilon} \leq -\gamma/2$ on $\partial V^{-1}(c)$, where $\dot{V}_{(8.7)_\epsilon}$ denotes the derivative of V along the solutions of $(8.7)_\epsilon$. Therefore, the bounded set $V^{-1}(c)$ has an ω-limit set $\mathcal{A}_\epsilon$ in $V^{-1}(c)$ with respect to the flow defined by $(8.7)_\epsilon$ and it is a local attractor for $V^{-1}(c)$. It remains to show the property of upper semicontinuity. For any $\eta < c$, there is a $t_0 = t_0(c, \eta)$ such that, for any $\xi \in V^{-1}(c)$, the solution $\varphi_0^t(\xi)$ of $(8.7)_0$ belongs to $V^{-1}(\eta/2)$ for $t \geq t_0$. Choose $\bar{\eta} = \bar{\eta}(\eta) > 0$ so that $\bar{\xi} \in V^{-1}(\eta)$ if $\xi \in V^{-1}(\eta/2)$ and $|\bar{\xi} - \xi| < \bar{\eta}$. We can find a $\delta_\eta > 0$ such that, for $0 \leq |\epsilon| < \delta_\eta$, the solution φ_ϵ^t of $(8.7)_\epsilon$ satisfies $|\varphi_\epsilon^t(\xi) - \varphi_0^t(\xi)| < \bar{\eta}$ for $\xi \in V^{-1}(c)$, $0 \leq t \leq t_0$. Since $\varphi_0^t(\xi) \in V^{-1}(\eta/2)$ for all $t \geq t_0$, this implies that $\varphi_\epsilon^t(\xi) \in V^{-1}(\eta)$ for $\xi \in V^{-1}(c)$, $t \geq t_0$. Therefore, $\mathcal{A}_\epsilon \subset V^{-1}(\eta)$ for $0 \leq |\epsilon| < \min\{\delta, \delta_\eta\}$. Since η is an arbitrary positive number, we may let $\eta \to 0$ and obtain the upper semicontinuity of the local attractors.

Exercise 8.7. Prove theorem 8.8 without using the converse theorem of Liapunov.

Let $\mathcal{A}_\epsilon$ be the local attractors given in Theorem 8.8. We say that the family of sets $\{\mathcal{A}_\epsilon,\ o \leq |\epsilon| < \delta\}$ is lower semicontinuous at $\epsilon = 0$ if

$$\lim_{\epsilon \to 0} \text{dist}\,(\mathcal{A}_0, \mathcal{A}_\epsilon) = 0.$$

Without further hypotheses about the flow on the attractor $\mathcal{A}_0$, we may not have lower semicontinuity. In fact, consider the simple example

$$\dot{x} = -(x+1)(x^2 + |\epsilon|).$$

If $\epsilon = 0$, the attractor is the line segment $[-1, 0]$. If $|\epsilon| \neq 0$, the attractor is the singleton $\{-1\}$.

Exercise 8.8. For a scalar differential equation depending continuously upon a parameter ϵ, give conditions on the vector field at $\epsilon = 0$ which will ensure that local attractors are lower semicontinuous at $\epsilon = 0$.

Exercise 8.9. Suppose that 0 is a local attractor for the equation $\dot{x} = f(x)$ and consider the perturbed equation

$$\dot{x} = f(x) + g(x). \tag{8.8}$$

Use Theorem 8.7 to conclude that, in a neighborhood of the origin, there is a positive constant K such that

$$\dot{V}_{(8.8)}(x) \leq -\alpha V(x) + K|g(x)|. \tag{8.9}$$

Use this result to give another proof of Theorem 8.8 for the case where the local attractor $\mathcal{A}$ consists only of the origin.

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