2.3. Stability.

In this section, we discuss stability properties of solutions of (1.1) in terms of a fundamental matrix solution of (1.1). The concepts of stability are the same for linear systems as for nonlinear systems. As a consequence, we first present the definitions for a nonlinear system.

Consider the d-dimensional system

$$\dot{x} = f(t, x) \tag{3.1}$$

and assume that $f(t, 0) = 0$ for all $t \geq 0$.

We say that the solution $x = 0$ is *stable* on $[0, \infty)$ if, for every $\epsilon > 0$, $\tau \geq 0$, there exists a $\delta = \delta(\epsilon, \tau) > 0$ such that $|\xi| < \delta$ implies that $|x(t, \tau, \xi)| < \epsilon$ for all $t \geq \tau$. We say that $x = 0$ is *uniformly stable* on $[0, \infty)$ if it is stable and $\delta = \delta(\epsilon)$ can be chosen independently of $\tau \geq 0$. We say that $x = 0$ is *asymptotically stable* on $[0, \infty)$ if it is stable and there exists a $b(\tau) > 0$ such that $|\xi| < b(\tau)$ implies that $|x(t, \tau, \xi)| \to 0$ as $t \to \infty$. We say that $x = 0$ is *uniformly asymptotically stable* on $[0, \infty)$ if it is stable and there exists a $b > 0$ such that, for every $\eta > 0$, there exists a $T(\eta) > 0$ such that, for any $\tau \geq 0$, $|\xi| < b$ implies that $|x(t, \tau, \xi)| < \eta$ for all $t \geq \tau + T(\eta)$.

If $\varphi(t)$ is an arbitrary solution of (3.1) defined on $[0, \infty)$, then the transformation $x = \varphi + y$ leads to a differential equation for y of the form $\dot{y} = g(t, y)$ where $g(t, 0) = 0$. We define the stability of φ in terms of the stability of the solution $y = 0$ of this equation.

Remark 3.1. For autonomous equations, $\dot{x} = f(x)$, the above definitions of stability are appropriate for the equilibrium solutions; that is, the solutions of $f(x) = 0$. The definitions are not appropriate for other types of invariant sets of autonomous systems. We will return to this in a later chapter, but it is instructive to think some at this time about the case where the invariant set is a close curve Γ which corresponds to a periodic solution $\varphi(t)$ of the equation. Since the system is autonomous, for any constant α, the function $\varphi(t + \alpha)$ also a periodic solution corresponding with the same orbit Γ. If α is small, then the difference $\varphi(t) - \varphi(t + \alpha)$ remains small for all t. Therefore, there is a type of stability if we consider the solutions that lie on Γ. However, if we take the initial value close to Γ but not on γ, then one does not expect in general that this solution will remain close to $\varphi(t + \alpha)$ for some α. This is easily seen from the pendulum equation $\ddot{x} + \sin x = 0$ if we note that the period of periodic solutions vary with the amplitude. This means that the definitions of stability must be modified in some appropriate way. It is instructive for the reader at this point to attempt to come up with a meaningful definition.

We have the following characterization of the concepts of stability for linear systems.

Theorem 3.1. *If $X(t)$ is a fundamental solution of (1.1), then*

(i) $x = 0$ *is stable on* $[0, \infty)$ *if and only if there exists a* $k = k(\tau) > 0$ *such that, for* $t \geq \tau \geq 0$, *we have* $|X(t)| \leq k$.

(ii) $x = 0$ *is uniformly stable on* $[0, \infty)$ *if and only if there exists a* $k > 0$ *such that, for* $t \geq s \geq 0$, *we have* $|X(t)X^{-1}(s)| \leq k$.

(iii) $x = 0$ *is asymptotically stable on* $[0, \infty)$ *if and only if* $|X(t)| \to 0$ *as* $t \to \infty$.

(iv) $x = 0$ *is uniformly asymptotically stable on* $[0, \infty)$ *if and only if there exists a* $k > 0$, $\alpha > 0$ *such that, for* $t \geq s \geq 0$, *we have* $|X(t)X^{-1}(s)| \leq ke^{-\alpha(t-s)}$.

Proof. In all cases, the "if" part of the theorem is easy if we observe that $x(t, \tau, \xi) = X(t)X^{-1}(\tau)\xi$. Therefore, we give the proof only for the "only if" part.

(i) If $x = 0$ is stable on $[0, \infty)$, then, for every $\epsilon > 0$, $\tau \geq 0$, there exists a $\delta = \delta(\epsilon, \tau) > 0$ such that $|\xi| < \delta$ implies that $|x(t, \tau, \xi)| < \epsilon$ for all $t \geq \tau$. Thus, $|X(t)X^{-1}(\tau)\xi| \leq \epsilon$ if $|\xi| \leq \delta$. This implies that

$$|X(t)X^{-1}(\tau)| = \sup_{|\eta| \leq 1} |X(t)X^{-1}(\tau)\eta| = \sup_{|\xi| \leq \delta} |X(t)X^{-1}(\tau)\frac{\xi}{\delta}| \leq \delta^{-1}\epsilon.$$

(ii) The proof is the same as in (i) with the observation that δ can be chosen independently of τ.

(iii) This is a consequence of the definition.

(iv) From (ii), there exists a $k_1 > 0$ such that $|X(t)X^{-1}(s)| \leq k_1$ for $t \geq s \geq 0$. Also, there exists a $b > 0$ such that, for every $\eta > 0$, there exists a $T = T(\eta) > 0$ such that, for any $s \geq 0$, $|\xi| < b$ implies that $|X(t)X^{-1}(s)\xi| < \eta$ for all $t \geq s + T(\eta)$. Therefore, $|X(t)X^{-1}(s)| \leq \frac{\eta}{b}$. Let $\alpha = -\frac{1}{T}\log\frac{\eta}{b}$ and $k = k_1 e^{\alpha T}$. For any $t \geq s \geq 0$, there exists an integer $n \geq 0$ such that $nT \leq t - s < (n+1)T$. With this notation, we have

$$\begin{aligned}
|X(t)X^{-1}(s)| &\leq |X(t)X^{-1}(s+nT)|\;|X(s+nT)X^{-1}(s)| \\
&\leq k_1|X(s+nT)X^{-1}(s+(n-1)T)|\;|X(s+(n-1)T)X^{-1}(s)| \\
&\leq k_1\frac{\eta}{b}|X(s+(n-1)T)X^{-1}(s+(n-2)T)|\;|X(s+(n-2)T)X^{-1}(s)| \\
&\cdots \\
&\leq k_1(\frac{\eta}{b})^n = k_1 e^{-\alpha nT} = ke^{-\alpha(n+1)T} \leq ke^{-\alpha(t-s)}\,.
\end{aligned}$$

This completes the proof of the theorem.

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