You can use your book and notes. No laptop or wireless devices allowed. Write clearly and try to make your arguments as linear and simple as possible. The complete solution of one exercise will be considered more that two half solutions.

There are 8 questions each worth from 20 to 30 point. A total of 100 point obtained through *completely* solved question will be considered a perfect score.

When returning your Exam, you must return also this page, signed. Thanks.

To solve the Exam problems, I have not collaborated with anyone or used any source except class notes and the textbook.

Name: ____________________

1. (20 points) Consider the differential equation

$$\dot{x} = F(x)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1 function. Assume that there exist an open simply connected region Ω and a function $g : \Omega \to \mathbb{R}$ such that $\nabla \cdot (g(x)F(x)) > 0$ for $x \in \Omega$. Show that there can be no periodic orbit completely contained in Ω.

Use the result to show that the equation

$$\begin{cases} \dot{x}_1 = & x_1(3x_1 - 4x_2 + 3) \\ \dot{x}_2 = & x_2(x_1^2 + 3x_2 + x_2^3) \end{cases}$$

has no periodic orbit completely contained in the region $\mathbb{R}^{2+} = \{(x_1, x_2) \,|\, x_1 > 0,\, x_2 > 0\}$.

2. Let $H(q, q)$ an Hamiltonian function of the form

$$H(q, p) = \frac{p^2}{2} + V(q)$$

where $p, q \in \mathbb{R}$. Assume that γ is a periodic orbit of the Hamilton equations genereated by H.

(a) (15 points) Show that γ cannot be asymptotically stable in the sense of section 5.3 of the book.

(b) (15 points) Let $x(t, x_0)$ be the soltuion of the Hamilton equation with $x(0, x_0) = x_0$, where $x = (q, p)$ and $x_0 = (q_0, p_0)$. Show that, if $x(t, x_0)$ is bounded and not periodic than $\lim_{t\to\infty} x(t, x_0)$ exists and is a fixed point.

3. (20 points) Consider the initial value problem

$$\begin{cases} \dot{x} = x^2 \cos(t) - e^t x \\ x(0) = x_0 \end{cases}$$

For which value of x_0, if any, is the solution defined for all $t \geq 0$?

4. (20 points) Let $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$ and $g(x_1, x_2) = (g_1(x_1, x_2), g_2(x_1, x_2))$ be two C^1 function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Assume that

$$f_1(x_1, x_2)g_2(x_1, x_2) - g_1(x_1, x_2)f_2(x_1, x_2) \geq 0 \qquad \forall (x_1, x_2) \in \mathbb{R}^2.$$

Shot that if the differential equation $\dot{x} = f(x)$ admit a periodic orbit than the equation $\dot{x} = g(x)$ has a fixed point.

5. (25 points) Consider the differential equation

$$\begin{cases} \dot{x} = F(x,t) \\ x(0) = x_0 \end{cases}$$

where $x \in \mathbb{R}^n$ and F is continuous in both x and t. Assume that there exists K such that

$$\|F(x,t) - F(y,t)\| \leq K\|x - y\| \qquad \forall x, y \in \mathbb{R}^n, \ \forall t \in \mathbb{R}.$$

Show that for every x_0 there is a unique solution and that such solution is defined for every t.

6. (20 points) Are there values of μ for which the system of differential equations

$$\begin{cases} \dot{x} = \mu x - y + xy - xy^2 - x^3 \\ \dot{y} = x + \mu y - yx^2 - y^3 \end{cases}$$

admit a non trivial periodic orbit? (**Bonus (up to 10pt)**: Characterize as well as you can the set of μ for which a periodic orbit exists.)

7. (20 points) Consider the system of differential equations

$$\begin{cases} \dot{x} = -x^3 - xy^2 \\ \dot{y} = -2x^2y - y^3 \end{cases}.$$

Show that the origin is asymptotically stable and globally attracting. Hint: look for a simple Lyapunov function.

8. (25 points) Let $N(a, b, T)$ the number of zeros in the interval $[0, T]$ of the solution of the differential equation

$$\ddot{x} + x \cos\left(\frac{t^4 + 3t^3 - 5t + 1}{2t^4 - 5t^2 + 6t - 4}\right) = 0$$

with initial condition $x(0) = a$ and $\dot{x}(0) = b$. Assume that $a^2 + b^2 \neq 0$. Compute

$$\lim_{T\to\infty} \frac{N(a, b, T)}{T} \, .$$