You can use your book and notes. No laptop or wireless devices allowed. Write clearly and try to make your arguments as linear and simple as possible. The complete solution of one exercise will be considered more that two half solutions.

Name: ______________________________

Question:	1	2	3	Total
Points:	50	40	20	110
Score:				

1. Consider the system of equation

$$\frac{dx}{dt} = ay + (x^2 - y^2) \tag{1}$$
$$\frac{dy}{dt} = -ax - 2xy \ . \tag{2}$$

(a) (10 points) Show that the function

$$H(x, y) = \frac{a}{2}\left(x^2 + y^2\right) + \left(x^2 y - \frac{y^3}{3}\right)$$

is a first integral (conserved quantity).

Solution: To be a first integral we need that

$$\partial_x H(x, y)\dot{x} + \partial_y H(x, y)\dot{y} = 0$$

and we get

$$(ax + 2xy)(ay + x^2 - y^2) + (ay + x^2 - y^2)(-ax - 2xy) = 0 \ .$$

Observe that

$$\frac{dx}{dt} = ay + (x^2 - y^2) = \partial_y H(x, y) \qquad \frac{dy}{dt} = -ax - 2xy = -\partial_x H(x, y)$$

so that the system is Hamiltonian.

(b) (20 points) Find all fixed point when $a \neq 0$.

Solution: From $\dot{y} = 0$ we get that either $x = 0$ or $y = -a/2$. Using $x = 0$ in $\dot{x} = 0$ we get either $y = 0$ or $y = a$ as solution. For $y = -a/2$ we get $x = \pm a\sqrt{3}/2$. Thus there are four fixed points:

$$(x_0, y_0) = (0, 0) \quad , \quad (x_1, y_1) = (0, a) \quad \text{and} \quad (x_2, \pm y_2) = \left(\pm\frac{\sqrt{3}}{2}a, -\frac{a}{2}\right)$$

(c) (20 points) For $a \neq 0$, derive the linearized system valid near each of the fixed point. Discuss their linear stability. What can you say on their nonlionear stability? (Can you apply the theorems in section 3.3 of the text book?)

Solution: For (x_0, y_0) the linerization is

$$\frac{dx}{dt} = ay \tag{3}$$
$$\frac{dy}{dt} = -ax \ . \tag{4}$$

that is a center.
For (x_1, y_1) the linerization is

$$\frac{dx}{dt} = -a(y-a) \tag{5}$$
$$\frac{d(y-a)}{dt} = -3ax \ . \tag{6}$$

Since the eigenvalue are $\lambda_{\pm} = \pm\sqrt{3}|a|$ we have that this is a saddle.
Finally for $(x_2, \pm y_2)$ we get

$$\frac{d}{dt}\left(x \mp \frac{\sqrt{3}}{2}a\right) = \pm\sqrt{3}a\left(x \mp \frac{\sqrt{3}}{2}a\right) + 2a\left(y - \frac{a}{2}\right) \tag{7}$$
$$\frac{d}{dt}\left(y + \frac{a}{2}\right) = \mp\sqrt{3}a\left(y - \frac{a}{2}\right) \ . \tag{8}$$

and also in this case we get that the eigenvalue are $\lambda_{\pm} = \pm\sqrt{3}|a|$ so also these points are saddles.
For the nonlinear stability we cannot say anything for $(0,0)$ using the theorems in section 3.3. The fact that the system is Hamiltonian and $(0,0)$ is a local minimum for H tell has that $(0,0)$ is still a center for the nonlinear system.
In the case of $(0,a)$ and $\left(-\frac{a}{2}, \pm\frac{\sqrt{3}}{2}a\right)$ we can say that they remain locally saddles. This means that, close to the fixed point, there is a stable and an unstable curve.

2. Consider the system

$$\frac{dx}{dt} = x - y + x^2 - 2x^3 - 2xy^2 \tag{9}$$
$$\frac{dy}{dt} = x + y + xy - 2y^3 - 2x^2y\ . \tag{10}$$

To answer the following question you may consider passing in polar coordinates.

Solution: We first convert the system in polar coordinates. We have

$$r\dot{r} = x\dot{x} + y\dot{y} = x^2 - xy + x^3 - 2x^4 - 2x^2y^2 + xy + y^2 + xy^2 - 2y^4 - 2x^2y^2 = \tag{11}$$
$$= r^2 + r^3(\sin\theta + \cos\theta) - 2r^4 \tag{12}$$

while clearly $\dot{\theta} = 1$. Thus we have

$$\frac{dr}{dt} = r + r^2\cos\theta - 2r^3 \tag{13}$$
$$\frac{d\theta}{dt} = 1. \tag{14}$$

(a) (20 points) Show that $(x, y) = (0, 0)$ is the unique fixed point for the system.

Solution: Clearly $(x, y) = (0, 0)$ is a fixed point and the equation for $\dot{\theta}$ tell us that there cann't be any other fixed point.

(b) (20 points) Use the Poincarè-Bendixon Thoerem to show that the system admit at least one periodic orbit.

Solution: Clearly $(0, 0)$ is a negative attractor so that the set $|(x, y)| > \epsilon$ is positively invariant for some small ϵ.
On the other hand, $r + r^2\cos\theta - 2r^3 \le r + r^2 - 2r^3$ so that $\dot{r} < 0$ if $r > 1$.
We can now apply the Poincaré-Bendixon Theorem to the annulus $\epsilon < r < 1$ and find a periodic ordbit.

3. (20 points) Consider a system of the form

$$\dot{x} = f(x)$$

with $x \in \mathbb{R}^n$, f continuous and differentiable for every x, and $f(0) = 0$, that is $x = 0$ is a fixed point. Show that if $x = 0$ is negatively asymptotically stable, then it cannot be positively stable.

Solution: Since $x = 0$ is negatively asymptotically stable we have that for x_0 small the solution $x(t, x_0)$ starting from x_0 converge to 0 when $t \to -\infty$. Thus for every δ there exists $t(\delta) > 0$ such that $|x(-t(\delta), x_0)| < \delta$. Calling $x(\delta) = x(-t(\delta), x_0)$ we have that $x(t(\delta), x(\delta)) = x_0$ that is if we choose $\epsilon < |x_0|$ for every δ there exists $x(\delta)$ with $|x(t, x(\delta))| > \epsilon$ for some t.