MATH 6337 EXAM 2 November 1, 2007

PLEASE READ THESE DIRECTIONS: Answer PROBLEM 1 (11 points) and choose TWO other problems to answer (12 points each). You may also answer (for up to 3 points extra credit) ONE additional problem. In this case, please specify which problem is the extra credit problem.

All statements require proof or justification. There are 35 points total, plus up to 3 points of extra credit.

Throughout this exam, you should assume that scalars are REAL unless specified otherwise.

1. Given $f \in L^1[0,1]$ (under Lebesgue measure), define $g(x) = \int_x^1 \frac{f(t)}{t}\, dt$. Show that g is defined a.e., that $g \in L^1[0,1]$, and that $\int_0^1 g(x)\, dx = \int_0^1 f(x)\, dx$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < \infty$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions, and let f be a measurable function. Prove that

$$f_n \xrightarrow{\mathrm{m}} f \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} \int_X \frac{|f - f_n|}{1 + |f - f_n|}\, d\mu = 0.$$

3. Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < \infty$. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions that are finite a.e., and f is a measurable function such that $f_n(x) \to f(x)$ pointwise a.e. Prove that there exist disjoint measurable sets $E_0, E_1, \dots$ with $\cup E_k = X$ such that $\mu(E_0) = 0$, and for each fixed $k \in \mathbb{N}$ we have that $f_n \to f$ uniformly on E_k as $n \to \infty$.

4. Scalars in this problem are complex. Let $f,\, g \in L^1(\mathbb{R})$ be given (Lebesgue measure).

(a) Prove that $(f * g)(x) = \int f(y)\, g(x-y)\, dy$ is measurable and $f * g \in L^1(\mathbb{R})$. You may assume without proof that $f(y)\, g(x-y)$ is measurable on $\mathbb{R}^2$.

(b) We proved in Homework 6 that $\hat{f}(\xi) = \int f(x)\, e^{-2\pi i \xi x}\, dx$ is defined for every ξ, and that $\hat{f}$ is a continuous and bounded function. Prove that

$$(f * g)^\wedge(\xi) \;=\; \hat{f}(\xi)\, \hat{g}(\xi), \quad \xi \in \mathbb{R}.$$

5. Suppose that $f \colon [0,1] \to [0,\infty]$ belongs to $L^1[0,1]$ (Lebesgue measure) and that for each $n \in \mathbb{N}$ we have

$$\int_0^1 f(x)^n\, dx \;=\; \int_0^1 f(x)\, dx.$$

Prove that there exists a measurable set $E \subseteq [0,1]$ such that $f = \chi_E$ a.e.