You can use your book and notes. No laptop or wireless devices allowed. Write clearly and try to make your arguments as linear and simple as possible. The complete solution of one exercise will be considered more that two half solutions. In your solution you can use only statements that were proven in class.

Name: ______________________________

Question:	1	2	3	4	Total
Points:	10	25	30	40	105
Score:					

1. (10 points) Let $f(x)$ be a bounded measurable function from $\mathbb{R}^n$ to $\mathbb{R}$ such that

$$\lim_{R\to\infty} \sup_{\|x\|>R} |f(x)|\|x\|^\alpha = C < \infty$$

where $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$. For which values of α can you say that $f \in L^1(\mathbb{R}^n)$? Is your condition on α necessary?

Solution: Since f is bounded we have that f is in $L^1(B_R)$ where $B_R = \{x \mid \|x\| \leq R\}$. Form the property of f we know that there exists R such that

$$\sup_{\|x\|>R} |f(x)|\|x\|^\alpha \leq C + 1$$

that is

$$|f(x)| \leq \frac{C+1}{\|x\|^\alpha} \qquad \text{for } \|x\| > R$$

From Corollary 2.52 in the book we know that if $\alpha > n$ then f is in $L^1(B_R^c)$. Thus $\alpha > n$ implies that $f \in L^1(\mathbb{R}^n)$.

Take now f such that $f(x) = 1$ if $i \leq \|x\| \leq i + 2^{-i}$, for every integer i. Clearly we have

$$\lim_{R\to\infty} \sup_{\|x\|>R} |f(x)| = 1$$

while

$$\lim_{R\to\infty} \sup_{\|x\|>R} |f(x)|\|x\|^\alpha = \infty$$

for every $\alpha > 0$. Notwithstanding this, $f \in L^1(\mathbb{R}^n)$.

2. Given $f \in L^1(\mathbb{R})$ define $F(x) = \int_{-\infty}^{x} f(y)dy$.

   (a) (10 points) Show that $F(x)$ is a continuous function. Is $F(x)$ differentiable?

Solution: Let y_n be such that $\lim_{n\to\infty} y_n = y$. Call $f_n(x) = \chi_{\{x\le y_n\}}(x)f(x)$. Clearly

$$\lim_{n\to\infty} f_n(x) = f(x)\chi_{\{x\le y\}}(x) \qquad \text{a.e.}$$

while $|f_n| \le |f|$. Thus

$$\lim_{n\to\infty} F(y_n) = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} f(x)\chi_{\{x\le y\}}(x)dx = F(y)$$

so that F is continuous.

Clearly if f is discontinuous in x, $F'(x)$ does not exists.

(b) (15 points) Show that for every $g \in C_0^1(\mathbb{R})$ we have:

$$\int_{\mathbb{R}} F(x)g'(x)dx = -\int_{\mathbb{R}} f(x)g(x)dx.$$

Remember that $C_0^1(\mathbb{R})$ is the set of all function in $C^1(\mathbb{R})$ that are 0 outside a compact set. Moreover $C^1(\mathbb{R})$ is the set of all function that are continuous and admit a continuous derivative.

Solution: From the definition we know that there are K, M such that $g(x) = g'(x) = 0$ for $|x| > K$ and $|g'(x)| < M$ for $|x| \leq K$. Let

$$h(x,t) = f(t)g'(x)\chi_{\{t<x\}}(x,t)$$

We have

$$\int_{\mathbb{R}^2} |h(x,t)|dxdt \leq \int_{\mathbb{R}^2} |f(t)||g'(x)|dxdt \leq 2KM \int_{\mathbb{R}} |f(t)|dt < \infty$$

Thus $h \in L^1(\mathbb{R}^2)$. From Tonelli Theorem we get

$$\int_{\mathbb{R}} F(x)g'(x)dx = \int_{\mathbb{R}^2} h(x,t)dxdt = \int_{\mathbb{R}} \int_t^{\infty} g'(x)f(t)dt = -\int_{\mathbb{R}} f(t)g(t)dt.$$

3. Let $f(x)$ be in $L^1(\mathbb{R})$ and let

$$\hat{f}(k) = \int_{\mathbb{R}} e^{ikx} f(x) dx.$$

(a) (15 points) Assume that $xf(x)$ is in $L^1(\mathbb{R})$. Show that the derivative $\hat{f}'(k)$ of $f(k)$ exists and is continuous. Moreover

$$\hat{f}'(k) = i \int_{\mathbb{R}} e^{ikx} x f(x) dx.$$

Solution: Let h_n be such that $h_n \to 0$ as $n \to \infty$. We have

$$\hat{f}(k + h_n) - \hat{f}(k) = \int_{\mathbb{R}} (e^{ih_n x} - 1) e^{ikx} f(x) dx$$

We first observe that, $|(e^{ih_n x} - 1) e^{ikx}| \leq 2$ so that, by the Dominated Convergence Theorem, we have that $\hat{f}(k)$ is continuous. Moreover

$$|e^{ix} - 1| \leq |x|$$

so that, for every n,

$$\left| \frac{e^{ih_n x} - 1}{h_n} e^{ikx} f(x) \right| \leq |x||f(x)|$$

Since $xf(x)$ is in $L^1(\mathbb{R})$ the Dominated Convergence Theorem implyes that:

$$\hat{f}(k) = \int_{\mathbb{R}} \frac{d}{dx} e^{ikx} f(x) dx = i \int_{\mathbb{R}} e^{ikx} x f(x) dx.$$

Finally reasoniong as above we get that $f'(k)$ is continuous.

(b) (15 points) Assume that $e^{\lambda|x|}f(x)$ is in $L^1(\mathbb{R})$. Show that, for h small enough, we have

$$\hat{f}(k+h) = \sum_{k=0}^{\infty} \frac{h^n}{n!} \hat{f}^{(n)}(k).$$

How big can h be?

Solution: We have

$$\hat{f}(k+h) = \int_{\mathbb{R}} e^{ihx} e^{ikx} f(x) dx = \int_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{h^n (ix)^n}{n!} e^{ikx} f(x) dx$$

Observe that

$$\left| \sum_{n=0}^{\infty} \frac{h^n (ix)^n}{n!} e^{ikx} f(x) \right| \leq |f(x)| \sum_{n=0}^{\infty} \frac{h^n |x|^n}{n!} = e^{h|x|} |f(x)|$$

If $h \leq \lambda$ we can apply again Dominated Convergence and obtain

$$\hat{f}(k+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} \int_{\mathbb{R}} (ix)^n e^{ikx} f(x) dx$$

Clearly we have that $x^n f(x)$ is in $L^1(\mathbb{R})$ so that iterating point a) we get

$$\int_{\mathbb{R}} (ix)^n e^{ikx} f(x) dx = \hat{f}^{(n)}(k).$$

4. Let f_n, $n = 1, 2, \ldots$, and f be measurable functions from a measure space $(X, \mathcal{M}, \mu)$ to $\mathbb{R}$ and let $h_n(x) = \max\{f_n(x), 0\}$ and $h(x) = \max\{f(x), 0\}$.

   (a) (10 points) Assume that $f_n \to f$ μ-a.e. Show $h_n \to h$ μ-a.e.

   Solution: Let x be such that $f_n(x) \to f(x)$. If $f(x) > 0$ then $f_n(x)$ is definitively positive and thus $h_n(x) = f_n(x) \to f(x) = h(x)$. If $f(x) < 0$ then $f_n(x) < 0$ definitively so that $h_n(x) = h(x) = 0$. Finally if $f(x) = 0$ then $f_n(x) \to 0$ and clearly $h_n(x) \to 0$.

(b) (15 points) Assume that $f_n \to f$ in $L^1(X, \mu)$. Show $h_n \to h$ in $L^1(X, \mu)$. (**Hint**: let $A_n = \{x \mid f_n(x) > 0\}$ and $A = \{x \mid f(x) > 0\}$. Write $X = (A \cap A_n) \cup (A^c \cap A_n^c) \cup (A^c \cap A_n) \cup (A \cap A_n^c)$. Convergence in the first two set is trivial. For the other two use the inequalities that define A and A_n.)

Solution: Observe that on $A \cap A_n$ we have $|h_n - h| = |f_n - f|$, on $A^c \cap A_n^c$ we have $|h_n - h| = 0$, on $A^c \cap A_n$ we have $|h_n - h| = h_n \leq f_n - f = |f_n - f|$ and finally on $A \cap A_n^c$ we have $|h_n - h| = h \leq f - f_n = |f_n - f|$. Thus for every $x \in X$ we have

$$|h_n - h| \leq |f_n - f|.$$

Thus if $f_n \to f$ in $L^1(X, \mu)$ then

$$\int_X |h_n - h| d\mu \leq \int_X |f_n - f| d\mu \to 0$$

(c) (15 points) Assume that $f_n \to f$ in measure. Show $h_n \to h$ in measure. (**Hint**: see the hint to point b).)

Solution: It follows immediately from

$$\{x \,|\, |h_n - h| > \epsilon\} \subset \{x \,|\, |f_n - f| > \epsilon\}.$$