n 6

a) LeT $E_i$ a finite (resp. countable) family of set in $\mathcal{R}$. Then

$$F = \bigcup_i E_i \in \mathcal{R}$$

Moreover

$$F / E_i \in \mathcal{R}$$

so That

$$\bigcap_i E_i = F / \bigcup_i (F/E_i) \in \mathcal{R}$$

b) if $X \in \mathcal{R}$ and $E \in \mathcal{R}$ then

$$X / E = E^c \in \mathcal{R}$$

c) LeT $\mathcal{A} = \{ E \subset X \mid E \in \mathcal{R} \text{ or } E^c \in \mathcal{R} \}$.

If $E_i \in \mathcal{A} \Rightarrow E_i \in \mathcal{R}$ or $E_i \notin \mathcal{R}$.

$$\bigcup_i E_i = \bigcup_{\substack{i \\ E_i \in \mathcal{R}}} E_i \cup \bigcup_{\substack{i \\ E_i^c \in \mathcal{R}}} E_i$$

buT $\left( \bigcup_{\substack{i \\ E_i^c \in \mathcal{R}}} E_i \right)^c = \bigcap_{\substack{i \\ E_i^c \in \mathcal{R}}} E_i^c \in \mathcal{R}$ by b)

~~Thus~~

Finally if

$$E \in R \quad \text{and} \quad F^c \in R$$

$$E \cup F = (F^c / E)^c \in A$$

So That

$$\bigcup_i E_i = \bigcup_{\substack{i \\ E_i \in R}} E_i \cup \bigcup_{\substack{i \\ E_i^c \in R}} E_i \in A$$

Clearly if $E \in A$ Then $E^c \in A$.

d) Let $A = \{ E \subset X \mid E \cap F \in R \text{ for all } F \in R \}$.

If $E \in A$

$$E^c \cap F = F \setminus (E \cap F)$$

but $E \cap F \in R$ so $E^c \cap F \in R$ and $E^c \in A$.

If $E_i \in A$ and $F \in R$ then

$$\bigcup_i E_i \cap F = \bigcup_i (E_i \cap F) \in R$$

since $E_i \cap F \in R$. So $\bigcup_i E_i \in A$

So $A$ is a $\sigma$ algebra

4.

If $\mathcal{A}$ is a $\sigma$ algebra Then $\mathcal{A}$ is closed under countable inceasing union.

Viceversa if $\mathcal{A}$ is closed under countable increasing union and $E_i \in \mathcal{A}$ Then

$$\bigcup_i E_i = \bigcup_i \left( \bigcup_{K=1}^{i} E_K \right) \in \mathcal{A}$$

becouse $\bigcup_{K=1}^{i} E_K \in \mathcal{A}$ ($\mathcal{A}$ is an algebra)

$$\bigcup_{K=1}^{i} E_K \subset \bigcup_{K=1}^{J} E_K \qquad i \leq J$$

Thus $\mathcal{A}$ is closed under countable union.

5) LeT $\mathcal{N} = \bigcup_{\substack{F \subset \mathcal{E} \\ F \text{ countable}}} \mathcal{M}(F)$

1) $\mathcal{N} \supset \mathcal{E}$ evidenT

2) $\mathcal{N}$ is a $\sigma$ algebra.

LeT $E_i \in \mathcal{N}$ Then $E_i \in \mathcal{M}(F_i)$ for some $F_i \subset \mathcal{E}$ and countable.

LeT $F = \bigcup_i F_i$ Then $F$ is counTable and $E_i \in \mathcal{M}(F)$ $\forall i$ so ThaT $\bigcup_i E_i \in \mathcal{M}(F) \subset \mathcal{N}$.

Thus $\mathcal{M}(\mathcal{E}) \subset \mathcal{N}$. The other inclusion is evident.

8. $$\mu(\liminf E_i) = \mu\left(\bigcup_{K=1}^{\infty} \bigcap_{i=K}^{\infty} E_i\right) =$$

observe $\bigcap_{i=K}^{\infty} E_i \supset \bigcap_{i=\ell}^{\infty} E_i \qquad K > \ell$

$$\mu(\liminf E_i) = \lim_{K\to\infty} \mu\left(\bigcap_{i=K}^{\infty} E_i\right)$$

Now
$$\mu\left(\bigcap_{K=i}^{\infty} E_K\right) \leq \inf_{K\geq i} \mu(E_K)$$

because $\bigcap_{K=i}^{\infty} E_K \subset E_K \quad \forall K$

Thus
$$\mu(\liminf E_i) \leq \lim_{K\to\infty} \inf_{i\geq K} \mu(E_i) =$$

$$= \liminf \mu(E_i)$$

On the other side if $F = \bigcup_{i=1}^{\infty} E_i$

$$F / \limsup E_i = F / \bigcap_{K=1}^{\infty} \bigcup_{i=K}^{\infty} E_i = \bigcup_{K=1}^{\infty} \bigcap_{i=K}^{\infty} F/E_i =$$

$\liminf F/E_i$

If $\mu(F) < +\infty$ Then

$$\mu(\liminf F/E_i) \le \liminf \mu(F/E_i) = \mu(F) + \liminf(-\mu(E_i))$$

The Thesis follow from

$$\liminf(-a_i) = -\limsup a_i$$

ii) If $\mu$ is continuous from below and $E_i$ are measurable and disjoint Then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty}\left(\bigcup_{k=1}^{i} E_k\right)\right) \underset{\text{continity}}{=} \lim_{i\to\infty} \mu\left(\bigcup_{k=1}^{i} E_k\right) =$$

$$= \lim_{i\to\infty} \sum_{k=1}^{i} \mu(E_k) = \sum_{k=1}^{\infty} \mu(E_k)$$

~~If $\mu$ is continuos from above and $E_i$ are measurable Let $F_i = F/E_i$ where $F = \bigcup_i E_i$. $\mu(F) \le \mu(X) < +\infty$ Thus~~

~~$\mu(\bigcup_i E_i) = \mu(F) - \mu(\bigcap_i F_i)$~~

If $\mu$ is continuous from above and $\mu(X) < +\infty$

$$\mu\left(\bigcup_i E_i\right) = \mu(X) - \mu\left(\bigcap_i E_i^c\right) =$$

$$\mu(X) - \mu\left(\bigcap_i \bigcap_{k \leq i} E_k^c\right) = \mu(X) - \lim_i \mu\left(\bigcap_{k \leq i} E_k^c\right)$$

The Thesis follow from the fact that

$$\mu\left(\bigcap_{k \leq i} E_k^c\right) = \mu(X) - \sum_{k=1}^{i} \mu(E_k)$$

16.

a) Let $X_i$ be such that

$\mu(X_i) < +\infty \qquad \bigcup_i X_i = X$

Then $E = \bigcup_i (E \cap X_i)$.

But $E \cap X_i \in \mathcal{M}$ since $\mu(X_i) < +\infty$ thus $E \in \mathcal{M}$.

b) Let $E_i \in \tilde{\mathcal{M}}$ and $A \in \mathcal{M}$ $\mu(A) < +\infty$

$$\bigcup_i E_i \cap A = \bigcup_i (E_i \cap A) \in \mathcal{M}$$

So $\bigcup_i E_i \in \tilde{\mathcal{M}}$

If $E \in \tilde{\mathcal{M}}$ and $A \in \mathcal{M}$

Then

$$E^c \cap A = (E \cap A)^c \cap A \in \mathcal{M}$$

c) We have to show that $\tilde{\mu}$ is a measure

Let $E_i \in \tilde{\mathcal{M}}$ disjoint and $F = \bigcup_i E_i$.

If $E_i \in \mathcal{M}$ $\forall i \Rightarrow \sum_i \tilde{\mu}(E_i) = \sum_i \mu(E_i) = \mu(F) = \tilde{\mu}(F)$

If $E_i \notin \mathcal{M}$ for some $i$ then $\tilde{\mu}(F) = +\infty$.

Indeed if $\tilde{\mu}(F) < \infty \Rightarrow F \in \mathcal{M}$ and $E_i \cap F = E_i \in \mathcal{M}$.

d) If $E \in \tilde{\mathcal{M}}$ and $\tilde{\mu}(E) = 0$ then $E \in \mathcal{M}$.

If $F \subset E$ then $F \subset \mathcal{M}$ and $\mu(F) = 0$ due to the completeness of $\mu$.

e) Clearly $\underline{\mu}$ is saturated.

Let $E_i \in \tilde{\mathcal{M}}$ and disjoint. If $A \subset \bigcup_i E_i$ and $\mu(A) < \infty$ then $A_i = A \cap E_i \in \mathcal{M}$.

Thus

$$\underline{\mu}\left(\bigcup_i E_i\right) = \sup_{A \subset \bigcup_i E_i} \mu(A) = \sup_{A \subset \bigcup_i E_i} \sum_i \mu(E_i \cap A)$$

$$\leq \sum_i \sup_{A_i \in E_i} \mu(A_i) = \sum_i \underline{\mu}(A_i)$$

Observe That if $A \subset \bigcup_i E_i$ and $\mu(A) = +\infty$ the for every $C$ there is $B \subset A$ with $\mu(B) > C$ so That

$$\sum_i \mu(E_i) \geq \sum_i \mu(E_i \cap B) = \mu(B) = C$$

Thus $\sum_i \mu(E_i) = +\infty$

The existence of $B$ is easily proved has follows. ~~Suppose $B \subset A$ such~~

Let $m = \sup_{\substack{B \subset A \\ \mu(B) < \infty}} \mu(B)$. If $m = +\infty$ we done. If $m < +\infty$ Take a sequence of $B_i$ such That $\mu(B_i) \geq m - 2^{-i}$.

~~$\bar{B}_k = \bigcup_{i=1}^{k} B_i$. If $\mu(\bar{B}) = +\infty$~~

~~Then~~ Call $\bar{B} = \bigcup_i B_i$.

Then $m - 2^{-i} \leq \mu(\bar{B}) \leq m$ Thus $\mu(\bar{B}) = m$

But There must be $C \subset A / \bar{B}$ with $0 < \mu(C) < +\infty$. Thus $\mu(\bar{B} \cup C) > m$ contradicting The hypothesis.

Finally if $E_i \in \tilde{\mathcal{M}}$ let $A_i$ be such that

$$\underline{\mu}(E_i) \le \mu(A_i) + \epsilon/2^i \quad \text{then}$$

$$\underline{\mu}(\cup E_i) \geq \mu(\bigcup_i A_i) = \sum_i \mu(A_i) \geq \sum_i \mu(E_i) - \epsilon$$

Thus $\underline{\mu}(\cup E_i) \geq \sum_i \mu(E_i)$.

[illegible]) If $A \subset X$ ~~and~~, $A$ is measurable and $\mu(A) < \infty$ then $A \cap X_1$ is finite. Thus $X/A \supset X_2/A$ is uncountable so that $A$ has to be countable.
If $E \subset X$ then $E \cap A$ is countable for every $A$ with $\mu(A) < +\infty$. Thus $E \cap A \in \mathcal{M}$ so that $\tilde{\mathcal{M}} = \mathcal{P}(X)$. It is easy to show that $\tilde{\mu}(X_2) = +\infty$ since $X_2^* \notin \mathcal{M}$.
On the other hand $\underline{\mu}(X_2) = 0$ because if $A \in \mathcal{M}$ and $A \subset X_2$ then $\mu(A) = 0$.

Finally if $E_i \in \tilde{\mathcal{M}}$ let $A_i$ be such that

$$\underline{\mu}(E_i) \le \mu(A_i) + \epsilon/2^i \quad \text{then}$$

$$\underline{\mu}(\cup E_i) \geq \mu(\bigcup_i A_i) = \sum_i \mu(A_i) \geq \sum_i \mu(E_i) - \epsilon$$

Thus $\underline{\mu}(\cup E_i) \geq \sum_i \mu(E_i)$.

b) If $A \subset X$ ~~and~~, $A$ is measurable and $\mu(A) < \infty$ then $A \cap X_1$ is finite. Thus $X/A \supset X_2/A$ is uncountable so that $A$ has to be countable. If $E \subset X$ then $E \cap A$ is countable for every $A$ with $\mu(A) < +\infty$. Thus $E \cap A \in \mathcal{M}$ so that $\tilde{\mathcal{M}} = \mathcal{P}(X)$. It is easy to show that $\tilde{\mu}(X_2) = +\infty$ since $X_2^* \notin \mathcal{M}$. On the other hand $\underline{\mu}(X_2) = 0$ because if $A \in \mathcal{M}$ and $A \subset X_2$ then $\mu(A) = 0$.