No books or notes allowed. No laptop or wireless devices allowed. Write clearly.

Name: ______________________________

Question:	1	2	3	4	Total
Points:	20	25	25	30	100
Score:					

Question 1 . . . . *20 point*

Use the Divergence Theorem to find the outward flux

$$\iint_S (\mathbf{F} \cdot \mathbf{n})\ dS$$

where

$$\mathbf{F}(x, y, z) = 2x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$$

and S is the suface of the sphere $x^2 + y^2 + z^2 = 2$.

Solution: We have

$$\nabla\mathbf{F} = 5$$

so that

$$\iint_S (\mathbf{F} \cdot \mathbf{n})\ dS = \iiint_D 5\, dV = 5\mathrm{Vol}(D) = 5\frac{4}{3}\pi\sqrt{2}^3 = \frac{40}{3}\sqrt{2}\pi$$

Question 2 . . . . *25 point*

Compute the Fourier series of the function $f(x)$ given by

$$f(x) = \begin{cases} x+1 & -1 < x \leq 0 \\ -x+1 & 0 < x \leq 1 \end{cases}$$

Solution: We want to write f as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

Since f is an even function we have

$$b_n = 0 \qquad \forall n$$

Moreover

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}$$

and

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = 2\int_0^1 (1-x)\cos(n\pi x)\,dx = \tag{1}$$

$$= 2\int_0^1 \cos(n\pi x)\,dx - 2\int_0^1 x\cos(n\pi x)\,dx \tag{2}$$

$$= 2\frac{1-\cos(n\pi)}{n^2\pi^2} \tag{3}$$

Question 3 . . . . *25 point*

Let $f(x)$ be a function with fourier series

$$f(x) = \sum_{n=1}^{\infty} a_n \cos(nx).$$

Show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \sum_{n=1}^{\infty} a_n^2$$

(**Hint:** substitute f with it Fourier series in the integral. Expand the square and use orthogonality)

Solution: We have

$$f(x)^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \cos(nx)\cos(mx)$$

so that

$$\int_{-\pi}^{\pi} f(x)^2\,dx = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \tag{4}$$

$$= \sum_{n=1}^{\infty} a_n^2 \int_{-\pi}^{\pi} \cos^2(nx)\,dx = \pi\sum_{n=1}^{\infty} a_n^2 \tag{5}$$

Question 4 . . . . *30 point*

Find the solution of the equation

$$\begin{cases} \Delta u(x,y) = 0 & (x,y) \in [0,\pi] \times [0,\pi] \\ u(0,y) = 0, \quad u(\pi,y) = 0 \\ u(x,0) = 0, \quad \dfrac{\partial u}{\partial y}(x,\pi) = f(x) \end{cases}$$

where $f(x)$ is a continuous function from $[0,\pi]$ to $\mathbb{R}$.

Solution: The homogeneous boundary conditions are

$$u(0,y) = 0, \quad u(\pi,y) = 0$$

Thus writing $u(x,y) = X(x)Y(y)$ and using separation of variables, we get

$$\begin{cases} X''(x) = -\lambda X(x) \quad X(0) = X(\pi) = 0 \\ Y''(y) = \lambda Y(y) \end{cases}$$

The first equation admits non trivial solutions only for $\lambda > 0$. Writing

$$X(x) = a\cos\left(\sqrt{\lambda}x\right) + b\sin\left(\sqrt{\lambda}x\right)$$

and imposing the boundary conditions we get

$$a = 0, \qquad \lambda = n^2.$$

The equation for Y gives

$$Y(y) = a\cosh\left(\sqrt{\lambda}y\right) + b\sinh\left(\sqrt{\lambda}y\right)$$

so that

$$u(x,y) = \sum_{n=1}^{\infty} \sin(nx)\left(a_n \cosh(ny) + b_n \sinh(ny)\right)$$

Setting $y = 0$ we get

$$\sum_{n=1}^{\infty} \sin(nx) a_n = 0$$

so that $a_n = 0$ for every n. Thus

$$\partial_y u(x,\pi) = \sum_{n=1}^{\infty} n b_n \sin(nx)\cosh(n\pi) = f(x)$$

so that

$$n b_n \cosh(n\pi) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)$$

Collecting everything we get

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin(nx) \sinh(ny)$$

where

$$b_n = \frac{2}{\pi} \frac{1}{n \cosh(n\pi)} \int_0^{\pi} f(x) \sin(nx)$$

Useful Formulas

Differential Operators.

$$\nabla f = (\partial_x f,\, \partial_y f,\, \partial_z f) \qquad \Delta u = \partial_x^2 u + \partial_y^2 u$$

Divergence Theorem.

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \iiint_D \nabla\cdot\mathbf{F}\, dV$$

where D is a region in $\mathbb{R}^3$ bounded by the surface S.

Trigonometry.

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)) \qquad \int x\cos(\alpha x)\, dx = \frac{\cos(\alpha x) + \alpha x\sin(\alpha x)}{\alpha^2}$$

Fourier Series: full period. If $f(x)$ is defined in $[-a, a]$ then its Fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{a}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{a}x\right)$$

Fourier Series: halph period. If $f(x)$ is defined in $[0, a]$ then its sine Fourier series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{a}x\right)$$

Potential Equation. To solve a potential equation first find all solutions of the form $u_n(x, y) = X_n(x)Y_n(y)$ for the equation with the homogeneous boundary conditions. Use the superposition principle to write a generic solution. Impose the remaining boundary condition and use orthogonality to compute the coefficients.