

# Section 1.2

## Row Reduction

## Review from 1.1: Solving Systems of Equations

### Example

Solve the system of equations

$$\begin{aligned}x + 2y + 3z &= 6 \\2x - 3y + 2z &= 14 \\3x + y - z &= -2\end{aligned}$$

This is the kind of problem we'll talk about for the first half of the course.

- ▶ A **solution** is a list of numbers  $x, y, z, \dots$  that makes *all* of the equations true.
- ▶ The **solution set** is the collection of all solutions.
- ▶ **Solving** the system means finding the solution set in a “parameterized” form.

### Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.

What is a *systematic* way to solve a system of equations?

## Review from 1.1: Solving Systems of Equations

### Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

What strategies do you know?

- ▶ Substitution
- ▶ Elimination

Both are perfectly valid, but only elimination scales well to large numbers of equations.

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

**Elimination method:** in what ways can you manipulate the equations?

- ▶ Multiply an equation by a nonzero number. (scale)
- ▶ Add a multiple of one equation to another. (replacement)
- ▶ Swap two equations. (swap)

# Solving Systems of Equations

Better notation

It sure is a pain to have to write  $x, y, z$ , and  $=$  over and over again.

**Matrix notation:** write just the numbers, in a box, instead!

$$\begin{array}{rcl} x + 2y + 3z = & 6 \\ 2x - 3y + 2z = & 14 \\ 3x + y - z = & -2 \end{array} \quad \begin{array}{l} \text{becomes} \\ \rightsquigarrow \end{array} \quad \left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

- ▶ Multiply all entries in a row by a nonzero number. **(scale)**
- ▶ Add a multiple of each entry of one row to the corresponding entry in another. **(row replacement)**
- ▶ Swap two rows. **(swap)**

### Important

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

### Definition

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

So the linear equations of row-equivalent matrices have the *same solution set*.

## Row Operations, Fundamental Example

### Example

Solve the system of equations

$$\begin{aligned}x + 2y + 3z &= 6 \\2x - 3y + 2z &= 14 \\3x + y - z &= -2\end{aligned}$$

Start:

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

**Goal:** we want our elimination method to eventually produce a system of equations like

$$\begin{array}{rcl} x & = & A \\ y & = & B \\ z & = & C \end{array} \quad \text{or in matrix form,} \quad \left( \begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array} \right)$$

So we need to do row operations that make the start matrix look like the end one.

**Strategy** (preliminary): fiddle with it so we only have ones and zeros. [\[animated\]](#)

# Row Operations

Continued

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

We want these to be zero.  
So we subtract multiples of the first row.

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array} \right)$$

We want these to be zero.

It would be nice if this were a 1.  
We could divide by  $-7$ , but that  
would produce ugly fractions.

Let's swap the last two rows first.

$$R_2 = R_2 - 2R_1$$

$$R_3 = R_3 - 3R_1$$

$$R_2 \leftrightarrow R_3$$

$$R_2 = R_2 \div -5$$

$$R_3 = R_3 + 7R_2$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array} \right)$$



# Row Operations

Continued

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array} \right)$$

$$\begin{array}{l} R_3 = R_3 \div 10 \\ \hline \end{array}$$

We want these to be zero.

$$\begin{array}{l} R_2 = R_2 - 2R_3, R_1 = R_1 - 3R_3 \\ \hline \end{array}$$

Let's make this a 1 first.

$$\begin{array}{l} R_1 = R_1 - 2R_2 \\ \hline \end{array}$$

translates into

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\begin{array}{rcl} x & = & 1 \\ y & = & -2 \\ z & = & 3 \end{array}$$

Success!

Check:

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

substitute solution

$$1 + 2 \cdot (-2) + 3 \cdot 3 = 6$$

$$2 \cdot 1 - 3 \cdot (-2) + 2 \cdot 3 = 14$$

$$3 \cdot 1 + (-2) - 3 = -2$$



## Row Echelon Form

Let's come up with an *algorithm* for turning an arbitrary matrix into a "solved" matrix. What do we mean by "solved"?

A matrix is in **row echelon form** if

1. All zero rows are at the bottom.
2. Each leading nonzero entry of a row is to the *right* of the leading entry of the row above.
3. Below a leading entry of a row, all entries are *zero*.

Picture:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\star$  = any number

$\boxed{\star}$  = any nonzero number

### Definition

A **pivot**  $\boxed{\star}$  is the first nonzero entry of a row of a matrix in row echelon form.

A **pivot column** is a column containing a pivot of a matrix in row echelon form.

## Reduced Row Echelon Form

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition,

4. The pivot in each nonzero row is equal to 1.
5. Each pivot is the only nonzero entry in its column.

Picture:

$$\begin{pmatrix} \mathbf{1} & 0 & * & 0 & * \\ 0 & \mathbf{1} & * & 0 & * \\ 0 & 0 & 0 & \mathbf{1} & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} * = \text{any number} \\ \mathbf{1} = \text{pivot} \end{array}$$

**Note:** Echelon forms do not care whether or not a column is augmented. Just ignore the vertical line.

### Question

Can every matrix be put into reduced row echelon form only using row operations?

**Answer:** Yes!

# Reduced Row Echelon Form

## Continued

Why is this the “solved” version of the matrix from the fundamental example?

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

It translates into

$$\begin{aligned} x &= 1 \\ y &= -2 \\ z &= 3, \end{aligned}$$

which is clearly the solution.

But what happens if there are fewer pivots than rows?

$$\left( \begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

... parametrized solution set (later).

# An Inconsistent Example

## Example

Solve the system of equations

$$x + y = 2$$

$$3x + 4y = 5$$

$$4x + 5y = 9$$

Let's try doing row operations: [\[interactive row reducer\]](#)

First clear these by subtracting multiples of the first row.

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array} \right) \begin{array}{l} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 4R_1 \end{array} \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 4 & 5 & 9 \end{array} \right)$$
$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array} \right)$$

Now clear this by subtracting the second row.

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array} \right) \begin{array}{l} R_3 = R_3 - R_2 \end{array} \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array} \right)$$

# An Inconsistent Example

Continued

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array} \right) \begin{array}{l} \text{translates into} \\ \text{~~~~~\rightsquigarrow} \end{array} \begin{array}{l} x + y = 2 \\ y = -1 \\ 0 = 2 \end{array}$$

In other words, the original equations

$$\begin{array}{l} x + y = 2 \\ 3x + 4y = 5 \\ 4x + 5y = 9 \end{array} \quad \text{have the same solutions as} \quad \begin{array}{l} x + y = 2 \\ y = -1 \\ 0 = 2 \end{array}$$

But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

In terms of the **augmented matrix**, what went wrong is that we had a **pivot in the rightmost column**. This means the side left of the equals sign is 0, but the right side is nonzero, which is impossible.

## Poll

Which of the following matrices are in reduced row echelon form?

A.  $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$       B.  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

C.  $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

D.  $(0 \ 1 \ 0 \ 0)$

E.  $(0 \ 1 \ 8 \ 0)$

F.  $\left( \begin{array}{cc|c} 1 & 17 & 0 \\ 0 & 0 & 1 \end{array} \right)$

**Answer:** B, D, E, F.

Note that A is in row echelon form though.

## Extra example 1

Translate the equation to an augmented matrix and put the matrix in RREF. Label all pivots. Feel free to use the [Interactive Row Reducer](#).

$$x_1 + 2x_2 + 2x_3 - x_4 = 4$$

$$2x_1 + 4x_2 + x_3 - 2x_4 = -1$$

$$-x_1 - 2x_2 - x_3 + x_4 = -1$$

$$\left( \begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ 2 & 4 & 1 & -2 & -1 \\ -1 & -2 & -1 & 1 & -1 \end{array} \right) \xrightarrow{\substack{R_2=R_2-2R_1 \\ R_3=R_3+R_1}} \left( \begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ 0 & 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & 0 & 3 \end{array} \right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left( \begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & -3 & 0 & -9 \end{array} \right)$$

$$\xrightarrow{\substack{R_3=R_3+3R_2 \\ R_1=R_1-2R_2}} \left( \begin{array}{cccc|c} \boxed{1} & 2 & 0 & -1 & -2 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$



## Extra example 2

Translate the equation to an augmented matrix and put the matrix in RREF. Label all pivots. Feel free to use the [Interactive Row Reducer](#).

$$x_3 + 3x_4 = 7$$

$$2x_1 - 6x_3 - 6x_4 = -6$$

$$4x_1 - 9x_3 - 3x_4 + x_5 = 8.$$

$$\left( \begin{array}{ccccc|c} 0 & 0 & 1 & 3 & 0 & 7 \\ 2 & 0 & -6 & -6 & 0 & -6 \\ 4 & 0 & -9 & -3 & 1 & 8 \end{array} \right) \xrightarrow{\text{work}} \left( \begin{array}{ccccc|c} \boxed{1} & 0 & 0 & 6 & 0 & 18 \\ 0 & 0 & \boxed{1} & 3 & 0 & 7 \\ 0 & 0 & 0 & 0 & \boxed{1} & -1 \end{array} \right)$$

## Summary

- ▶ We can more easily do elimination with matrices. The allowable moves are row swaps, row scales, and row replacements. This is called row reduction.
- ▶ A matrix in row echelon form corresponds to a system of linear equations that we can easily solve by back substitution.
- ▶ A matrix in reduced row echelon form corresponds to a system of linear equations that we can easily solve just by looking.
- ▶ We have an algorithm for row reducing a matrix to reduced row echelon form.
- ▶ The reduced row echelon form of a matrix is unique.
- ▶ Two matrices that differ by row operations are called row equivalent. Row-equivalent systems have the *same solution set*.
- ▶ A system of equations is inconsistent **exactly** when the corresponding augmented matrix has a pivot in the last column.