Supplemental problems: §5.4

1. True or false. Answer true if the statement is always true. Otherwise, answer false.

a) If A is an invertible matrix and A is diagonalizable, then A^{-1} is diagonalizable.

b) A diagonalizable $n \times n$ matrix admits n linearly independent eigenvectors.

c) If A is diagonalizable, then A has n distinct eigenvalues.

Solution.

a) True. If $A = PDP^{-1}$ and A is invertible then its eigenvalues are all nonzero, so the diagonal entries of D are nonzero and thus D is invertible (pivot in every diagonal position). Thus, $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$.

b) True. By the Diagonalization Theorem, an $n \times n$ matrix is diagonalizable *if and only if* it admits n linearly independent eigenvectors.

c) False. For instance, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal but has only one eigenvalue.

2. Give examples of 2×2 matrices with the following properties. Justify your answers.

a) A matrix A which is invertible and diagonalizable.

b) A matrix B which is invertible but not diagonalizable.

c) A matrix C which is not invertible but is diagonalizable.

d) A matrix D which is neither invertible nor diagonalizable.

Solution.

a) We can take any diagonal matrix with nonzero diagonal entries:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b) A shear has only one eigenvalue $\lambda = 1$. The associated eigenspace is the x-axis, so there do not exist two linearly independent eigenvectors. Hence it is not diagonalizable.

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

c) We can take any diagonal matrix with some zero diagonal entries:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

d) Such a matrix can only have the eigenvalue zero — otherwise it would have two eigenvalues, hence be diagonalizable. Thus the characteristic polynomial is $f(\lambda)=\lambda^2$. Here is a matrix with trace and determinant zero, whose zero-eigenspace (i.e., null space) is not all of $\mathbf{R}^2$:

$$D=\begin{pmatrix}0&1\\0&0\end{pmatrix}.$$

3. $A=\begin{pmatrix}2&3&1\\3&2&4\\0&0&-1\end{pmatrix}$.

a) Find the eigenvalues of A, and find a basis for each eigenspace.

b) Is A diagonalizable? If your answer is yes, find a diagonal matrix D and an invertible matrix C so that $A=CDC^{-1}$. If your answer is no, justify why A is not diagonalizable.

Solution.

a) We solve $0=\det(A-\lambda I)$.

$$0=\det\begin{pmatrix}2-\lambda&3&1\\3&2-\lambda&4\\0&0&-1-\lambda\end{pmatrix}=(-1-\lambda)(-1)^6\det\begin{pmatrix}2-\lambda&3\\3&2-\lambda\end{pmatrix}=(-1-\lambda)((2-\lambda)^2-9)$$

$$=(-1-\lambda)(\lambda^2-4\lambda-5)=-(\lambda+1)^2(\lambda-5).$$

So $\lambda=-1$ and $\lambda=5$ are the eigenvalues.

$\underline{\lambda=-1}$: $\big(A+I\mid 0\big)=\left(\begin{array}{ccc|c}3&3&1&0\\3&3&4&0\\0&0&0&0\end{array}\right)\xrightarrow{R_2=R_2-R_1}\left(\begin{array}{ccc|c}3&3&1&0\\0&0&1&0\\0&0&0&0\end{array}\right)\xrightarrow[\text{then } R_1=R_1/3]{R_1=R_1-R_2}$ $\left(\begin{array}{ccc|c}1&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right)$, with solution $x_1=-x_2$, $x_2=x_2$, $x_3=0$. The (-1)-eigenspace has basis $\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$.

$\underline{\lambda=5}$:

$$\big(A-5I\mid 0\big)=\left(\begin{array}{ccc|c}-3&3&1&0\\3&-3&4&0\\0&0&-6&0\end{array}\right)\xrightarrow[R_3=R_3/(-6)]{R_2=R_2+R_1}\left(\begin{array}{ccc|c}-3&3&1&0\\0&0&5&0\\0&0&1&0\end{array}\right)\xrightarrow[\text{then } R_2\leftrightarrow R_3,\ R_1/(-3)]{R_1=R_1-R_3,\ R_2=R_2-5R_3}\left(\begin{array}{ccc|c}1&-1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right),$$

with solution $x_1=x_2$, $x_2=x_2$, $x_3=0$. The 5-eigenspace has basis $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$.

b) A is a 3×3 matrix that only admits 2 linearly independent eigenvectors, so A is not diagonalizable.

4. Let $A = \begin{pmatrix} 8 & 36 & 62 \\ -6 & -34 & -62 \\ 3 & 18 & 33 \end{pmatrix}$.

The characteristic polynomial for A is $(\lambda-2)^2(\lambda-3)$. Decide if A is diagonalizable. If it is, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

Solution.

For $\lambda_1 = 3$, we row-reduce $A - 3I$:

$$\begin{pmatrix} 5 & 36 & 62 \\ -6 & -37 & -62 \\ 3 & 18 & 30 \end{pmatrix} \xrightarrow[\text{(New } R_1)/3]{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 6 & 10 \\ -6 & -37 & -62 \\ 5 & 36 & 62 \end{pmatrix} \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 + 6R_1} \begin{pmatrix} 1 & 6 & 10 \\ 0 & -1 & -2 \\ 0 & 6 & 12 \end{pmatrix}$$

$$\xrightarrow[\text{then } R_2 = -R_2]{R_3 = R_3 + 6R_2} \begin{pmatrix} 1 & 6 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 = R_1 - 6R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the solutions to $\left(A - 3I \mid 0\right)$ are $x_1 = 2x_3,\ x_2 = -2x_3,\ x_3 = x_3$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}. \qquad \text{The 3-eigenspace has basis } \left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

For $\lambda_2 = 2$, we row-reduce $A - 2I$:

$$\begin{pmatrix} 6 & 36 & 62 \\ -6 & -36 & -62 \\ 3 & 18 & 31 \end{pmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & 6 & \frac{31}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The solutions to $\left(A - 2I \quad 0\right)$ are $x_1 = -6x_2 - \frac{31}{3}x_3,\ x_2 = x_2,\ x_3 = x_3$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6x_2 - \frac{31}{3}x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix}.$$

The 2-eigenspace has basis $\left\{ \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix} \right\}$.

Therefore, $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 2 & -6 & -\frac{31}{3} \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note that we arranged the eigenvectors in C in order of the eigenvalues $3,\ 2,\ 2$, so we had to put the diagonals of D in the same order.

5. Which of the following 3×3 matrices are necessarily diagonalizable over the real numbers? (Circle all that apply.)

1. A matrix with three distinct real eigenvalues.
2. A matrix with one real eigenvalue.
3. A matrix with a real eigenvalue λ of algebraic multiplicity 2, such that the λ-eigenspace has dimension 2.
4. A matrix with a real eigenvalue λ such that the λ-eigenspace has dimension 2.

Solution.

The matrices in 1 and 3 are diagonalizable. A matrix with three distinct real eigenvalues automatically admits three linearly independent eigenvectors. If a matrix A has a real eigenvalue λ_1 of algebraic multiplicity 2, then it has another real eigenvalue λ_2 of algebraic multiplicity 1. The two eigenspaces provide three linearly independent eigenvectors.

The matrices in 2 and 4 need not be diagonalizable.

6. Suppose a 2×2 matrix A has eigenvalue $\lambda_1 = -2$ with eigenvector $v_1 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$, and eigenvalue $\lambda_2 = -1$ with eigenvector $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a) Find A.

b) Find A^{100}.

Solution.

a) We have $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

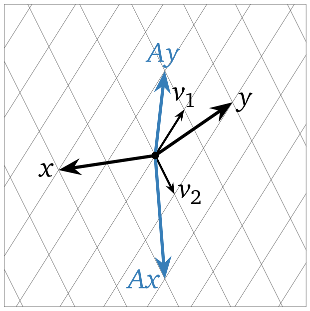
We compute $C^{-1} = \dfrac{1}{-5/2}\begin{pmatrix} -1 & -1 \\ -1 & 3/2 \end{pmatrix} = \dfrac{1}{5}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix}$.

$$A = CDC^{-1} = \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -8 & -3 \\ -2 & -7 \end{pmatrix}.$$

b)

$$\begin{aligned} A^{100} = CD^{100}C^{-1} &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \cdot D^{100}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2^{100} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \cdot 2^{100} & 2 \cdot 2^{100} \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3 \cdot 2^{100} + 2 & 3 \cdot 2^{100} - 3 \\ 2^{101} - 2 & 2^{101} + 3 \end{pmatrix}. \end{aligned}$$

7. Suppose that $A = C\begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}C^{-1}$, where C has columns v_1 and v_2. Given x and y in the picture below, draw the vectors Ax and Ay.



Solution.

A does the same thing as $D = \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}$, but in the v_1, v_2-coordinate system. Since D scales the first coordinate by $1/2$ and the second coordinate by -1, hence A scales the v_1-coordinate by $1/2$ and the v_2-coordinate by -1.

Supplemental problems: §5.5

1. **a)** If A is the matrix that implements rotation by 143° in $\mathbf{R}^2$, then A has no real eigenvalues.

b) A 3×3 matrix can have eigenvalues $3, 5$, and $2+i$.

c) If $v = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 1-i$, then $w = \begin{pmatrix} 2i-1 \\ i \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 1-i$.

Solution.

a) True. If A had a real eigenvalue λ, then we would have $Ax = \lambda x$ for some nonzero vector x in $\mathbf{R}^2$. This means that x would lie on the same line through the origin as the rotation of x by 143°, which is impossible.

b) False. If $2+i$ is an eigenvalue then so is its conjugate $2-i$.

c) True. Any nonzero complex multiple of v is also an eigenvector for eigenvalue $1-i$, and $w = iv$.

2. Consider the matrix

$$A = \begin{pmatrix} 3\sqrt{3}-1 & -5\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3}-1 \end{pmatrix}$$

a) Find both complex eigenvalues of A.

b) Find an eigenvector corresponding to each eigenvalue.

Solution.

a) We compute the characteristic polynomial:

$$\begin{aligned} f(\lambda) &= \det\begin{pmatrix} 3\sqrt{3}-1-\lambda & -5\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3}-1-\lambda \end{pmatrix} \\ &= (-1-\lambda+3\sqrt{3})(-1-\lambda-3\sqrt{3}) + (2)(5)(3) \\ &= (-1-\lambda)^2 - 9(3) + 10(3) \\ &= \lambda^2 + 2\lambda + 4. \end{aligned}$$

By the quadratic formula,

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(4)}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i.$$

b) Let $\lambda = -1 - \sqrt{3}i$. Then

$$A - \lambda I = \begin{pmatrix} (i+3)\sqrt{3} & -5\sqrt{3} \\ 2\sqrt{3} & (i-3)\sqrt{3} \end{pmatrix}.$$

Since $\det(A-\lambda I)=0$, the second row is a multiple of the first, so a row echelon form of A is

$$\begin{pmatrix} i+3 & -5 \\ 0 & 0 \end{pmatrix}.$$

Hence an eigenvector with eigenvalue $-1-\sqrt{3}i$ is $v=\begin{pmatrix} 5 \\ 3+i \end{pmatrix}$. It follows that an eigenvector with eigenvalue $-1+\sqrt{3}i$ is $\overline{v}=\begin{pmatrix} 5 \\ 3-i \end{pmatrix}$.

3. This problem shows an example of a matrix that has a mix of eigenvalues that are real and not real. It isn't computationally feasible on an exam, so doing this problem in full is just for fun. However, understanding the possibilities for eigenvalues of an $n\times n$ matrix in terms of the Fundamental Theorem of Algebra is a key component of section 5.5.

Let $A=\begin{pmatrix} 4 & -3 & 3 \\ 3 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$. Find all eigenvalues of A. For each eigenvalue of A, find a corresponding eigenvector.

Solution.

First we compute the characteristic polynomial by expanding cofactors along the third row:

$$\begin{aligned} f(\lambda) &= \det\begin{pmatrix} 4-\lambda & -3 & 3 \\ 3 & 4-\lambda & -2 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{pmatrix} \\ &= (2-\lambda)\big((4-\lambda)^2+9\big) = (2-\lambda)(\lambda^2-8\lambda+25). \end{aligned}$$

Using the quadratic equation on the second factor, we find the eigenvalues

$$\lambda_1 = 2 \qquad \lambda_2 = 4-3i \qquad \overline{\lambda}_2 = 4+3i.$$

Next compute an eigenvector with eigenvalue $\lambda_1=2$:

$$A-2I=\begin{pmatrix} 2 & -3 & 3 \\ 3 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form is $x=0$, $y=z$, so the parametric vector form of the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{eigenvector}} v_1=\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Now we compute an eigenvector with eigenvalue $\lambda_2 = 4-3i$:

$$A = (4-3i)I = \begin{pmatrix} 3i & -3 & 3 \\ 3 & 3i & -2 \\ 0 & 0 & 3i-2 \end{pmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{pmatrix} 3 & 3i & -2 \\ 3i & -3 & 3 \\ 0 & 0 & 3i-2 \end{pmatrix}$$

$$\xrightarrow{R_2=R_2-iR_1} \begin{pmatrix} 3 & 3i & -2 \\ 0 & 0 & 3+2i \\ 0 & 0 & 3i-2 \end{pmatrix} \xrightarrow{R_2=R_2\div(3+2i)} \begin{pmatrix} 3 & 3i & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 3i-2 \end{pmatrix}$$

$$\xrightarrow{\text{row replacements}} \begin{pmatrix} 3 & 3i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1=R_1\div 3} \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form of the solution is $x = -iy$, $z = 0$, so the parametric vector form is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \overset{\text{eigenvector}}{\rightsquigarrow} v_2 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}.$$

An eigenvector for the complex conjugate eigenvalue $\overline{\lambda}_2 = 4+3i$ is the complex conjugate eigenvector $\overline{v}_2 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$.