Math 1553 Worksheet §5.4, 5.5

1. True or false. If the statement is always true, answer true and justify why it is true. Otherwise, answer false and give an example that shows it is false. If not explicitly stated, assume A is an $n \times n$ matrix.

a) A 3×3 matrix A can have a non-real complex eigenvalue with multiplicity 2.

b) If A is the 3×3 the matrix for the orthogonal projection of vectors in $\mathbf{R}^3$ onto the plane $x + y + z = 0$, then A is diagonalizable.

c) If the RREF of A is diagonalizable, then A must be diagonalizable.

Solution.

a) No. If c is a (non-real) complex eigenvalue with multiplicity 2, then its conjugate $\bar{c}$ is an eigenvalue with multiplicity 2 since complex eigenvalues always occur in conjugate pairs. This would mean A has a characteristic polynomial of degree 4 or more, which is impossible since A is 3×3.

b) Yes, it is diagonalizable. The projection fixes all vectors in the plane $x+y+z = 0$ and destroys all vectors in the perpendicular line that goes through the origin, so its 1-eigenspace is two-dimensional and its 0-eigenspace is one-dimensional, thus we have a basis for $\mathbf{R}^3$ consisting of eigenvectors of A.

Alternative (but much more work): we could find three linearly independent eigenvectors by finding two linearly independent vectors satisfying $x+y+z = 0$ and then doing guess-and-check to find a vector that is perpendicular to both of them. For example, the vectors $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ span the 1-eigenspace, and $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ spans the 0-eigenspace.

c) No, for example, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable but its RREF is the identity matrix which is diagonalizable.

2. Suppose A is a 2×2 matrix satisfying

$$A\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}2\\-2\end{pmatrix},\qquad A\begin{pmatrix}-2\\3\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

a) Diagonalize A by finding 2×2 matrices C and D (with D diagonal) so that $A=CDC^{-1}$.

b) Find A^{17}.

Solution.

a) From the information given, $\lambda_1=-2$ is an eigenvalue for A with corresponding eigenvector $\begin{pmatrix}-1\\1\end{pmatrix}$, and $\lambda_2=0$ is an eigenvalue with eigenvector $\begin{pmatrix}-2\\3\end{pmatrix}$.

By the Diagonalization Theorem, $A=CDC^{-1}$ where

$$C=\begin{pmatrix}-1&-2\\1&3\end{pmatrix},\qquad D=\begin{pmatrix}-2&0\\0&0\end{pmatrix}.$$

b) We find $C^{-1}=\dfrac{1}{-3+2}\begin{pmatrix}3&2\\-1&-1\end{pmatrix}=\begin{pmatrix}-3&-2\\1&1\end{pmatrix}.$

$$\begin{aligned}A^{17}=CD^{17}C^{-1}&=\begin{pmatrix}-1&-2\\1&3\end{pmatrix}\begin{pmatrix}(-2)^{17}&0\\0&0\end{pmatrix}\begin{pmatrix}-3&-2\\1&1\end{pmatrix}\\&=\begin{pmatrix}-1&-2\\1&3\end{pmatrix}\begin{pmatrix}3\cdot 2^{17}&2^{18}\\0&0\end{pmatrix}\\&=\begin{pmatrix}-3\cdot 2^{17}&-2^{18}\\3\cdot 2^{17}&2^{18}\end{pmatrix}.\end{aligned}$$

3. Let $A=\begin{pmatrix}2&3\\-1&1\end{pmatrix}\begin{pmatrix}1&0\\0&1/2\end{pmatrix}\begin{pmatrix}2&3\\-1&1\end{pmatrix}^{-1}$, and let $x=\begin{pmatrix}2\\-1\end{pmatrix}+\begin{pmatrix}3\\1\end{pmatrix}$. What happens to $A^n x$ as n gets very large?

Solution.

We are given diagonalization of A, which gives us the eigenvalues and eigenvectors.

$$\begin{aligned}A^n x=A^n\left(\begin{pmatrix}2\\-1\end{pmatrix}+\begin{pmatrix}3\\1\end{pmatrix}\right)&=A^n\begin{pmatrix}2\\-1\end{pmatrix}+A^n\begin{pmatrix}3\\1\end{pmatrix}\\&=1^n\begin{pmatrix}2\\-1\end{pmatrix}+\left(\frac{1}{2}\right)^n\begin{pmatrix}3\\1\end{pmatrix}\\&=\begin{pmatrix}2\\-1\end{pmatrix}+\begin{pmatrix}\frac{3}{2^n}\\\frac{1}{2^n}\end{pmatrix}.\end{aligned}$$

As n gets very large, the entries in the second vector above approach zero, so $A^n x$ approaches $\begin{pmatrix}2\\-1\end{pmatrix}$. For example, for $n=15$,

$$A^{15}x\approx\begin{pmatrix}2.00009\\-0.999969\end{pmatrix}.$$

4. Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Find all eigenvalues of A. For each eigenvalue, find an associated eigenvector.

Solution.

The characteristic polynomial is

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 5$$

$$\lambda^2 - 2\lambda + 5 = 0 \iff \lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

For the eigenvalue $\lambda = 1-2i$, we use the shortcut trick you may have seen in class: the first row $\begin{pmatrix} a & b \end{pmatrix}$ of $A - \lambda I$ will lead to an eigenvector $\begin{pmatrix} -b \\ a \end{pmatrix}$ (or equivalently, $\begin{pmatrix} b \\ -a \end{pmatrix}$ if you prefer).

$$\left(A-(1-2i)I \mid 0 \right) = \left(\begin{array}{cc|c} 2i & 2 & 0 \\ (*) & (*) & 0 \end{array} \right) \implies v = \begin{pmatrix} -2 \\ 2i \end{pmatrix}.$$

From the correspondence between conjugate eigenvalues and their eigenvectors, we know (without doing any additional work!) that for the eigenvalue $\lambda = 1+2i$, a corresponding eigenvector is $w = \overline{v} = \begin{pmatrix} -2 \\ -2i \end{pmatrix}$.

If you used row-reduction for finding eigenvectors, you would find $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$ as an eigenvector for eigenvalue $1-2i$, and $w = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ as an eigenvector for eigenvalue $1+2i$.

5. Give an example of matrices A and B which have the same eigenvalues and the same algebraic multiplicities for each eigenvalue, so that A is diagonalizable but B is not diagonalizable. Justify your answer.

Solution.

Many examples possible. For example, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Both A and B have characteristic equation $\lambda^2 = 0$, so each has eigenvalue $\lambda = 0$ with algebraic multiplicity two. However, A is diagonalizable (in fact, diagonal!) but B is not.

Another possible example would be $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.