Math 1553 Practice Final Exam, Fall 2021

Solutions

1. Answer True or False to each of the following questions.

a) If $\{u, v, w\}$ is a linearly independent set of vectors in $\mathbf{R}^n$, then the dimension of Span$\{u, v, w\}$ must be 3.

b) If $\{u, v\}$ is a linearly independent set of vectors in $\mathbf{R}^5$ and $\{w, x\}$ is a linearly independent set of vectors in $\mathbf{R}^5$, then $\{u, v, w, x\}$ is linearly independent.

Solution.

a) True. We can see this using the Increasing Span Criterion or the Basis Theorem.

b) False. For example

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad x = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In this case, Span$\{u, v, w, x\}$ is only a plane.

2. Let $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x - y \geq 0 \right\}$. Answer yes or no to each of the following questions.

a) Is W closed under addition?

b) Is W close under scalar multiplication?

Solution.

a) Yes. We could see this geometrically by drawing the set W in the plane, which is all points satisfying $x \geq y$. To see it algebraically: If $\begin{pmatrix} x \\ y \end{pmatrix}$ is in W and $\begin{pmatrix} z \\ w \end{pmatrix}$ is in W, then $x \geq y$ and $z \geq q$, so for $\begin{pmatrix} x+z \\ y+w \end{pmatrix}$ we have

$$(x+z) - (y+w) = (x-y) + (z-w) \geq 0.$$

b) No. For example, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is in W but $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ is not since $-2 - (-1) < 0$.

3. Answer true or false to each of the following questions.

a) Suppose that A is a 3×7 matrix. Then the column space of A satisfies

$$\dim(\text{Col } A) \leq 3.$$

b) If A is a 2021×1553 matrix, then

$$\dim(\text{Col } A) + \dim(\text{Nul } A) = 1553.$$

Solution.

a) True. The column space of A is a subspace of $\mathbf{R}^3$, so its dimension is at most 3.

b) True, by the Rank Theorem.

4. Consider the matrix A and its reduced row echelon form below:

$$A = \begin{pmatrix} 1 & 1 & 2 & -2 \\ 1 & 0 & 1 & -2 \\ 2 & 1 & 3 & -4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Which of the following is a basis for Col(A)?

a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

b) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

c) $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\}$

d) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right\}$

Solution.

The pivot columns of A form a basis for Col(A), so a basis is

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

5. **a)** Consider the transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ given by

$$T(x_1, x_2, x_3, x_4) = (2x_4 - x_2,\ x_3 - x_1,\ x_1 - x_3).$$

Is T onto?
(i) Yes
(ii) No

b) Find the matrix A the reflects vectors in $\mathbf{R}^2$ across the line $y = x$.

(i) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(iii) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(iv) $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(v) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

(vi) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(vii) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution.

a) No. Every vector in the range of T has second coordinate equal to -1 times its third coordinate, so for example, $(0,1,0)$ is not in the range of T.

b) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

6. Let $A = \begin{pmatrix} 1 & 3 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Answer the following questions.

a) What is the dimension of Nul(A)?
(i) 1
(ii) 2
(iii) 3
(iv) 4
(v) 5

b) What is the dimension of Row(A)?
(i) 1
(ii) 2
(iii) 3
(iv) 4
(v) 5

Solution.

a) The homogeneous system $\left(A \mid 0 \right)$ has three free variables since A has 5 columns but 2 pivots, so Nul(A) has dimension 3.

b) $\dim(\text{Row}(A)) = 2$. We can see this either by noting that $\dim(\text{Row}(A)) = \dim(\text{Col}(A)) = 2$ or by using the fact $\text{Row}(A) = \text{Col}(A^T)$ and quickly finding a basis for $\text{Col}(A^T)$ consisting of two vectors.

7. In each case, determine whether the given set is a subspace of $\mathbf{R}^3$.

a) $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + 2y + 3z = 0 \right\}.$

b) $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + 2y + 3z = 4 \right\}.$

c) $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; y = x^2, z = x^3 \right\}.$

d) $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x = y = 0 \right\}.$

Solution.

a) Yes.

b) No. Doesn't contain 0.

c) No.

d) Yes. This V is just the z-axis in $\mathbf{R}^3$.

8. Let T be the linear transformation

$$T(x_1, x_2, x_3) = (0,\ 2x_1 - x_2,\ x_2 - 3x_3,\ x_1).$$

Find the standard matrix A for T. In other words, find the matrix A so that $T(x) = Ax$.

a) $\begin{pmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 2 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$

Solution.

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{pmatrix}$$

9. Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & k & -1 \end{pmatrix}$, and let T be the matrix transformation $T(x) = Ax$.

a) Find all values of k so that T is one-to-one.
(i) $k = 0$ only
(ii) $k = 1$ only
(iii) $k = -1$ only
(iv) $k = 2$ only
(v) $k = -2$ only
(vi) T is one-to-one unless $k = 0$
(vii) T is one-to-one unless $k = 2$
(viii) There is no value of k that makes T one-to-one.

b) Find all values of k so that T is onto.
(i) $k = 0$ only
(ii) $k = 1$ only
(iii) $k = -1$ only
(iv) $k = 2$ only
(v) $k = -2$ only
(vi) T is onto unless $k = 0$
(vii) T is onto unless $k = 2$
(viii) There is no value of k that makes T onto.

Solution.

a) There is no k so that T is one-to-one, since A has more columns than rows.

b) T is onto unless the second column is $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ (which would make all three columns to be scalar multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$), so the answer is all real k except $k = 2$.

10. **a)** True or false: If A and B are matrices and the products AB and BA are both defined, then A and B must be square matrices with the same number of rows and columns.

b) Suppose B is an $m \times n$ matrix and A is an $n \times p$ matrix.

If x is a vector in $\mathbf{R}^p$ and $BAx = 0$, then which one of the following **must** be true?
I. x must be in Nul(A).
II. Ax must be in Nul(B).
III. x must be in Nul(B).

Solution.

a) False. Taken almost verbatim from a problem in the 3.4 supplement.

b) II. We can write $BAx = 0$ as $B(Ax) = 0$, so Ax must be in the null space of B.
Note: I. is not the correct answer here! Sometimes x is in the null space of A but this is certainly not guaranteed. For example, take $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $BAx = 0$ but x is not in the null space of A.

11. Suppose A is a 5×5 matrix. Answer each of the following questions.

a) If every vector in $\mathbf{R}^5$ can be written as a linear combination of the columns of A, then A must be invertible.

b) If $Ax = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ has exactly one solution, then A must be invertible.

Solution.

a) True, by the Invertible Matrix Theorem.

b) True. If $Ax = b$ has exactly one solution for some b, then A must have a pivot in every column, so $Ax = 0$ has exactly one solution. Therefore, A is invertible by the IMT.

12. Suppose A is a 7×5 matrix with rank 5, and let T be the matrix transformation $T(x) = Ax$.

a) Which one of the following statements is true about the range of T?
I. The range of T is $\mathbf{R}^5$.
II. The range of T is a 5-dimensional subspace of $\mathbf{R}^7$.
III. The range of T has a basis consisting of 7 vectors.
IV. The range of T is $\mathbf{R}^7$.

b) Is T one-to-one?
I. Yes
II. No

Solution.

a) II. The range of T is the column space of A, which lives in $\mathbf{R}^7$ because A has 7 rows, and which has dimension 5 since the rank of A is 5.

b) Yes. A has 5 columns and has rank 5, so A has a pivot in every column and thus T is one-to-one.

13. Answer Yes, No, or Maybe to each of the following questions

a) Suppose A is an $n \times n$ invertible matrix. Is A^T invertible?

b) Suppose A is an $n \times n$ invertible matrix. Is $3A$ invertible?

Solution.

a) Yes, and in fact $(A^T)^{-1} = (A^{-1})^T$.

b) Yes, and in fact $(3A)^{-1} = \frac{1}{3}A^{-1}$.

14. Find the area of the triangle with vertices $(-1,2)$, $(0,5)$, and $(3,-4)$. Enter a single number as your answer. If your answer is not a whole number, enter a fraction as your answer.

Solution.

Many approaches are possible. We will use the vector from the first to the second point and the vector from the second to the third point, so the triangle is formed using the vectors $v_1 = (1,3)$ and $v_2 = (4,-6)$.

$$\text{Area} = \frac{1}{2}\left|\begin{pmatrix} 1 & 4 \\ 3 & -6 \end{pmatrix}\right| = \frac{1}{2}\Big| -6 - 12 \Big| = 9.$$

15. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, and let $B = \begin{pmatrix} c & c & c \\ 3 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, where c is a nonzero real number.

Which of the following must be correct?

I. $\det A = \frac{1}{c}\det B$
II. $\det A = c \det B$
III. $\det A = \det B$
IV. $\det A = c^3 \det B$
V. $\det A = c + \det B$

Solution.

I. $\det A = \frac{1}{c}\det B$. To get the matrix A using B, we multiply the first row of B by $1/c$, which multiplies the determinant of B by $1/c$.

16. Suppose A and B are 2×2 matrices with $\det(A) = 6$ and $\det(B) = 3$. Find $\det(-2AB^{-1})$. Enter a single number as your answer.

Solution.

$$\det(-2AB^{-1}) = (-2)^2 \det(AB^{-1}) = 4 \cdot \det(A) \cdot \frac{1}{\det(B)} = 4(6)(1/3) = 8.$$

17. Let $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$. Find $A^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

a) $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

b) $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

c) $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

d) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

e) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

f) $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

g) None of these

Solution.

$A^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is just the first column of A^{-1} by the definition of matrix multiplication, but we could also just compute A^{-1} and multiply by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ if we wish.

$$A^{-1} = \frac{1}{3-2}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}, \qquad A^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

18. Answer True or False to each of the following questions.

a) The matrix $A = \begin{pmatrix} \cos(47^\circ) & -\sin(47^\circ) \\ \sin(47^\circ) & \cos(47^\circ) \end{pmatrix}$ does **not** have any real eigenvalues.

b) Suppose A is a 3×3 matrix and that u and v are eigenvectors of A. Then $u + v$ is also an eigenvector of A.

Solution.

Both parts of this question were nearly copied from our third midterm or its practice exam.

a) True.

b) False. If u and v correspond to two different eigenvalues, then $u+v$ will not be an eigenvector. For example, if $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ then e_1 and e_2 are eigenvectors but $e_1 + e_2$ is not an eigenvector.

19. Find the value of k so that $\lambda = 5$ is an eigenvalue of the following matrix:

$$A = \begin{pmatrix} 6 & 1 & k \\ 2 & 6 & 3 \\ -1 & 0 & 4 \end{pmatrix}.$$

Enter a single number as your answer.

Solution.

We solve for $\det(A - 5I) = 0$ and get

$$0 = \det \begin{pmatrix} 1 & 1 & k \\ 2 & 1 & 3 \\ -1 & 0 & -1 \end{pmatrix} = 1(-1) - 1(-2+3) + k(0+1) = -2 + k,$$

so $k = 2$.

20. Let A be the 2×2 matrix that reflects vectors in $\mathbf{R}^2$ across the line $x_2 = -4x_1$. Which of the following vectors is an eigenvector of A corresponding to the eigenvalue $\lambda = -1$?

I. $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$

II. $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

III. $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$

IV. $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

V. $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

VI. $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

VII. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Solution.

This is a problem we have seen before two or three times.
A flips vectors perpendicular to the line $x_2 = -4x_1$, so we need a vector along the line $x_2 = x_1/4$. This vector is $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, so IV is the correct answer.

21. Find all values of c so that the following matrix has exactly one real eigenvalue with algebraic multiplicity 2.

$$A = \begin{pmatrix} 2 & 3 \\ c & -6 \end{pmatrix}$$

a) $c = 1$

b) $c = -1$

c) $c = 8/3$

d) $c = -8/3$

e) $c = 16/3$

f) $c = -16/3$

g) The matrix A will have two distinct eigenvalues, no matter what value of c we choose.

Solution.

The characteristic polynomial is

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 + 4\lambda - 12 - 3c.$$

For this to be a perfect square it must be $\lambda^2 + 4\lambda + 4$, so we set $-12 - 3c = 4$ and get $-3c = 16$, so $c = -16/3$.

22. Let A be a 3×3 matrix. Suppose that the null space of A is a line, and that $Ax = x$ for all x in the subspace W given below:

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 - x_3 = 0 \right\}.$$

Is A diagonalizable?

a) Yes, A must be diagonalizable.

b) No, A cannot be diagonalizable.

c) We need more information to determine whether A is diagonalizable.

Solution.

Yes. Its null space is a line (i.e. its 0-eigenspace is a line) and its 1-eigenspace is the plane W, so A has three linearly independent eigenvectors in $\mathbf{R}^3$ and is therefore diagonalizable.

23. Suppose A is a positive stochastic 3×3 matrix with the property that

$$\mathrm{Nul}(A - I) = \mathrm{Span}\left\{ \begin{pmatrix} 18 \\ 21 \\ 13 \end{pmatrix} \right\} \qquad \text{and} \qquad \mathrm{Nul}(A + \frac{3}{10}I) = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Answer the following questions.

a) What is the steady-state vector for A?

(i) $\begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$ (ii) $\begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}$ (iii) $\frac{1}{52}\begin{pmatrix} 18 \\ 21 \\ 13 \end{pmatrix}$ (iv) $\begin{pmatrix} 18/13 \\ 21/13 \\ 1 \end{pmatrix}$

(v) $\begin{pmatrix} 18 \\ 21 \\ 13 \end{pmatrix}$ (vi) We need more information.

b) What vector does $A^n \begin{pmatrix} 48 \\ 2 \\ 2 \end{pmatrix}$ approach as n gets very large?

(i) 0 (ii) ∞ (iii) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (iv) $\begin{pmatrix} 18 \\ 21 \\ 13 \end{pmatrix}$ (v) $\begin{pmatrix} 26 \\ 26 \\ 0 \end{pmatrix}$ (vi) $\begin{pmatrix} -26 \\ 26 \\ 2/3 \end{pmatrix}$

(vii) $\begin{pmatrix} 48 \\ 2 \\ 2 \end{pmatrix}$ (viii) $\begin{pmatrix} 18/13 \\ 21/13 \\ 1 \end{pmatrix}$

Solution.

a) The steady-state vector spans the 1-eigenspace, so we divide by spanning vector for Nul$(A-I)$ by the sum of its entries:

$$w = \frac{1}{18+21+13}\begin{pmatrix} 18 \\ 21 \\ 13 \end{pmatrix} = \frac{1}{52}\begin{pmatrix} 18 \\ 21 \\ 13 \end{pmatrix}.$$

b) $A^n v$ approaches cw where c is the sum of the entries of v. Here $c = 48+2+2 = 52$, so $A^n v$ approaches

$$52 \cdot \frac{1}{52}\begin{pmatrix} 18 \\ 21 \\ 13 \end{pmatrix} = \begin{pmatrix} 18 \\ 21 \\ 13 \end{pmatrix}.$$

24. Suppose A is a 2×2 matrix and its characteristic polynomial is

$$\det(A - \lambda I) = (2-\lambda)(3-\lambda).$$

What is the characteristic polynomial of A^2?

a) $(2-\lambda)^2(3-\lambda)^2$

b) $(2-\lambda)(3-\lambda)$

c) $(2-\lambda^2)(3-\lambda^2)$

d) $(4-\lambda)(9-\lambda)$

e) $(4-\lambda^2)(9-\lambda^2)$

Solution.

This one was taken from the reading day practice problems. The eigenvalues of A^2 are $2^2 = 4$ and $3^2 = 9$, and the degree of the characteristic polynomial for A^2 is 2 since A is a 2×2 matrix.

Therefore, $\det(A^2 - \lambda I) = (4-\lambda)(9-\lambda)$.

25. Suppose A is a 2×2 matrix whose entries are real numbers, and suppose that $\lambda_1 = 4 - 2i$ is an eigenvalue with corresponding eigenvector $v_1 = \begin{pmatrix} 7 \\ 3+2i \end{pmatrix}$.

a) (a) Which of the following is true?
(i) A must have another eigenvalue $\lambda_2 = 2-4i$.
(ii) A must have another eigenvalue $\lambda_2 = -4-2i$.
(iii) A must have another eigenvalue $\lambda_2 = 2+4i$.
(iv) A must have another eigenvalue $\lambda_2 = 4+2i$.
(v) It is possible that the eigenvalue $\lambda_1 = 4-2i$ has algebraic multiplicity 2.

b) (b) Which of the following vectors must be an eigenvector of A?
(i) $\begin{pmatrix} 7 \\ 3-2i \end{pmatrix}$
(ii) $\begin{pmatrix} 7 \\ -2-3i \end{pmatrix}$
(iii) $\begin{pmatrix} -3-2i \\ 7 \end{pmatrix}$
(iv) $\begin{pmatrix} -7 \\ 3+2i \end{pmatrix}$

Solution.

a) A must have second eigenvalue $\lambda_2 = \overline{\lambda_1} = 4+2i$.

b) We take the complex conjugate of each entry of v_1:

$$v_2 = \begin{pmatrix} 7 \\ 3-2i \end{pmatrix}.$$

26. Suppose A is a 5×5 matrix with characteristic polynomial

$$\det(A-\lambda I) = (2-\lambda)^3(4-\lambda)(5-\lambda).$$

a) Is A invertible?
(i) Yes, A must be invertible.
(ii) No, A is not invertible.
(iii) We need more information to determine whether A is invertible.

b) Suppose the 2-eigenspace of A is 3-dimensional. Is A diagonalizable?
(i) Yes, A is diagonalizable.
(ii) No, A is not diagonalizable.
(iii) We need more information to determine whether A is diagonalizable.

Solution.

a) Yes. $\det(A) = \det(A-0I) = (2^3)(4)(5) \neq 0$ so A is invertible.

b) Yes. In this case, $\lambda = 2$ has geometric multiplicity 3 while $\lambda = 4$ and $\lambda = 5$ are each guaranteed geometric multiplicity 1, so A has 5 linearly independent eigenvectors in $\mathbf{R}^5$ and is thus diagonalizable.

27. Let

$$A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}^{-1}$$

and let

$$x = \begin{pmatrix} 6 \\ 2 \end{pmatrix}.$$

What vector does $A^n x$ approach n gets very large?

a) $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

b) $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

c) $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

d) $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$

e) The entries of $A^n x$ keep getting larger and larger without approaching any vector as n gets large.

Solution.

This is a problem from a chapter 5 worksheet once we observe that $x = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

$$A^n x = A^n \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} \right) = A^n \begin{pmatrix} 2 \\ -1 \end{pmatrix} + A^n \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \left(\frac{1}{3} \right)^n \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

This approaches $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ as n gets very large.

28. Suppose u and v are orthogonal vectors in $\mathbf{R}^8$ satisfying $||u|| = 2$ and $||v|| = 1$. Compute the dot product below.

$$(2u + 3v) \cdot (4u - 5v).$$

Enter a single number as your answer.

Solution.

Similar to a reading day problem. By the properties of dot products,

$$\begin{aligned}(2u + 3v) \cdot (4u - 5v) &= 8(u \cdot u) - 10(u \cdot v) + 12(v \cdot u) - 15(v \cdot v) \\ &= 8||u||^2 - 10(0) + 12(0) - 15||v||^2 \; = \; 8(4) - 15(1) \; = \; 17.\end{aligned}$$

29. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Which of the following is a basis for $(\text{Nul}\, A)^{\perp}$?

I. $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$

II. $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}4\\5\\6\end{pmatrix}\right\}$

III. $\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$

IV. $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$

V. $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$

Solution.

Taken almost directly from the 6.2 Webwork. This problem can be done in many different ways. The fastest is to use facts about orthogonal complements:

$$(\operatorname{Nul} A)^\perp = \operatorname{Row} A = \operatorname{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}4\\5\\6\end{pmatrix}\right\},$$

so the answer is II.

Alternatively, we could have computed $\operatorname{Nul} A$ and then compute a basis for $(\operatorname{Nul} A)^\perp$ directly, then recognized that II is an equivalent basis. Even another alternative would be to compute that $\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$ is a basis for $\operatorname{Nul} A$ and then find that $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}4\\5\\6\end{pmatrix}$ are both orthogonal to $\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$.

The process of elimination could also be used to do this problem. It is quick to find that $\operatorname{Nul} A$ is a 1-dimensional subspace of $\mathbf{R}^3$, so its orthogonal complement must be a two-dimensional subspace of $\mathbf{R}^3$. This means that the only possibilities are II and IV, and if you find $\operatorname{Nul} A$ you will see IV is clearly incorrect.

30. **a)** Suppose W is a subspace of $\mathbf{R}^n$. If v is a vector in W, then the orthogonal projection of v onto W must be the zero vector.

b) If B is the matrix for the orthogonal projection onto a subspace W of $\mathbf{R}^n$, then $B^2 = B$.

Solution.

a) False. If v is in W, then the the orthogonal projection of v onto W is v itself.

b) True. This is a standard fact from section 6.3.

31. Let W be the subspace of $\mathbf{R}^4$ consisting of all vectors of the form $\begin{pmatrix} x \\ x+y \\ x+y+z \\ x+y \end{pmatrix}$,

where x, y, and z are real numbers. Find a basis for $W^\perp$.

a) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

b) $\left\{ \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

c) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}$

d) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

e) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Solution.

$W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$, so

$$W^\perp = \text{Nul}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The above matrix augmented with 0 row-reduces to $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ which gives $x_1 = 0$, $x_2 = -x_4$, $x_3 = 0$, and x_4 free. Thus a basis for Nul(A) is

$$\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Alternatively we could have just eyeballed the answer. It is clear that W is 3-dimensional so $W^{\perp}$ is 1-dimensional, and the vector $\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ is orthogonal to any vector in W because its dot product with any vector in W gives $-(x+y)+x+y=0$.

32. Let $W = \text{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 4 \\ -2 \end{pmatrix}\right\}$. Answer True or False for each question below.

a) The dimension of $W^{\perp}$ is 2.

b) The matrix B which implements orthogonal projection onto W is

$$B = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 1 & 4 \\ 1 & -2 \end{pmatrix}.$$

Solution.

a) True. $\dim(W) = 2$ and W is a subspace of $\mathbf{R}^4$, so $\dim(W^{\perp}) = 4-2 = 2$.

b) False. The matrix given in this part is not even a square matrix.

33. **a)** If u and v are **nonzero** orthogonal vectors in $\mathbf{R}^3$, then the set $\{u, v\}$ must be linearly independent.

b) Suppose A is an $m \times n$ matrix and b is a vector in Col (A). If $\widehat{x}$ is a least-squares solution to $Ax = b$, then $A\widehat{x} = b$.

Solution.

a) True. If u and v were nonzero orthogonal vectors that were linearly *dependent*, then $v = cu$ for some nonzero c and we would simultaneously have $u \cdot v = 0$ and

$$u \cdot v = u \cdot cu = c||u||^2 \neq 0.$$

which is impossible (it says $0 \neq 0$).

b) True. Taken almost verbatim from course material. If b is in the column space of A, then any "least squares solution" to $Ax = b$ is simply a "solution" to $Ax = b$ since $A\widehat{x} = b_{\text{Col}\,A} = b$.

34. Find the matrix for the orthogonal projection onto the line spanned by $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

a) $\begin{pmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{pmatrix}$

b) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

c) $\begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix}$

d) $\begin{pmatrix} 9/7 & 12/7 \\ 12/7 & 16/7 \end{pmatrix}$

e) $\begin{pmatrix} 9/5 & 12/5 \\ 12/5 & 16/5 \end{pmatrix}$

f) $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

Solution.

Let $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. The matrix in question is

$$\frac{1}{u \cdot u} u u^T = \frac{1}{9+16} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 3 & 4 \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}.$$

35. The least-squares solution for the matrix equation $\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$ is $\widehat{x} = \begin{pmatrix} a \\ b \end{pmatrix}$.

Find a.

a) $a = 0$

b) $a = 1$

c) $a = -1$

d) $a = 2/7$

e) $a = -17/5$

f) $a = 2$

g) $a = 1/7$

Solution.

We can actually do this problem without any work whatsoever! Note that $\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$ is the first column of $\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix}$, so $\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$ and we get $a = 1$ and $b = 0$ immediately.

Alternatively, we could do the standard direct computations. With $A=\begin{pmatrix}0&1\\1&1\\4&1\end{pmatrix}$, we solve $A^TA\widehat{x}=A^T\begin{pmatrix}0\\1\\4\end{pmatrix}$.

$$\begin{pmatrix}0&1&4\\1&1&1\end{pmatrix}\begin{pmatrix}0&1\\1&1\\4&1\end{pmatrix}\widehat{x}=\begin{pmatrix}0&1&4\\1&1&1\end{pmatrix}\begin{pmatrix}0\\1\\4\end{pmatrix}$$

$$\begin{pmatrix}17&5\\5&3\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}17\\5\end{pmatrix}.$$

Row-reducing $\left(\begin{array}{cc|c}17&5&17\\5&3&5\end{array}\right)$ gives $a=1$ and $b=0$, so $a=1$.

36. Let B be the matrix for orthogonal projection onto the subspace W given below: $W=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\text{ in }\mathbf{R}^3 \;\middle|\; x-y-2z=0\right\}$.

Which of the following is the 1-eigenspace of B?

I. $\mathrm{Nul}\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

II. $\mathrm{Nul}\begin{pmatrix}1&-1&-2\end{pmatrix}$

III. $\mathrm{Row}\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

IV. $\mathrm{Col}\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

V. $\mathrm{Row}\begin{pmatrix}1&2\\1&0\\0&1\end{pmatrix}$

Solution.

B destroys all vectors in $W^\perp$ and fixes all vectors in W. Its 1-eigenspace is equal to W, and $W=\mathrm{Nul}\begin{pmatrix}1&-1&-2\end{pmatrix}$. The answer is II.

37. The goal of this problem is to find A and b that will enable us to find the least-squares line $y=Mx+B$ that best fits the data $(-3,3)$, $(1,1)$, and $(4,7)$.

In other words, we need A and b that will enable us to find the least-squares solution to $A\begin{pmatrix}M\\B\end{pmatrix}=b$.

a) What is the matrix A in the equation $A\begin{pmatrix} M \\ B \end{pmatrix} = b$?

(i) $A = \begin{pmatrix} 1 & -3 \\ 1 & 1 \\ 1 & 4 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 3 & 1 \\ 1 & 1 \\ 7 & 1 \end{pmatrix}$ (iii) $A = \begin{pmatrix} -3 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix}$

(iv) $A = \begin{pmatrix} 1 & 1 & 1 \\ -3 & 1 & 4 \end{pmatrix}$ (v) $A = \begin{pmatrix} -3 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix}$ (vi) $A = \begin{pmatrix} 1 & 1 & 1 \\ -3 & 1 & 4 \end{pmatrix}$

(vii) $A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 7 \end{pmatrix}$

b) What is the vector b in the equation $A\begin{pmatrix} M \\ B \end{pmatrix} = b$?

(i) $b = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ (ii) $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (iii) $b = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$ (iv) $b = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$

(v) $b = \begin{pmatrix} -3 & 1 & 4 \end{pmatrix}$ (vi) $b = \begin{pmatrix} 3 & 1 & 7 \end{pmatrix}$

Solution.

We do both parts of this problem together. Note that the problem specifies the order of M and B where $y = Mx + B$ is the least-squares line, so we need to be careful with the ordering of our columns or we will mix up M and B.

$$x = -3, y = 3 : 3 = M(-3) + B$$
$$x = 1, y = 1 : 1 = M(1) + B$$
$$x = 4, y = 7 : 7 = M(4) + B$$

This gives us the system

$$-3M + B = 3$$
$$M + B = 1$$
$$4M + B = 7$$

which corresponds to the matrix equation

$$\begin{pmatrix} -3 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} M \\ B \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}.$$

Therefore, $A = \begin{pmatrix} -3 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$.

38. Suppose W is a subspace of $\mathbf{R}^3$ and x is a vector in $\mathbf{R}^3$ whose orthogonal decomposition with respect to W is $x = x_W + x_{W^\perp}$, where

$$x_W = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} \quad \text{and} \quad x_{W^\perp} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}.$$

a) Find the closest vector to x in W.

(i) $\begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$ (iii) $\begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix}$ (iv) $\begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}$ (v) $\begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$

b) Find the distance from x to W.

(i) 3 (ii) $\sqrt{12}$ (iii) $\sqrt{56}$ (iv) $\sqrt{14}$ (v) $\sqrt{42}$ (vi) $\sqrt{8}$

Solution.

a) The closest vector to x in W is x_W, which was given to us: $\begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}$.

b) The distance is

$$||x_{W^\perp}|| = \sqrt{1^2 + (-2)^2 + (-3)^2} = \sqrt{14}.$$

39. Find the orthogonal projection of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ onto the line spanned by $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

(i) $\begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix}$

(ii) $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

(iii) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(iv) 0

(v) $\begin{pmatrix} -8/5 \\ 6/5 \end{pmatrix}$

(vi) $-2/5$

Solution.

Let $u = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $x = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. We need the orthogonal projection of x onto the line spanned by u. We can use the formula from chapter 6 or just compute it directly:

$$\frac{1}{u \cdot u} uu^T x = \frac{1}{u \cdot u} u(u \cdot x) = \frac{u \cdot x}{u \cdot u} u = \frac{-10}{25} u = -\frac{2}{5} \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -8/5 \\ 6/5 \end{pmatrix}.$$

40. Suppose A is a 5×8 matrix with three pivots. Answer the following questions. In each case, enter a single number as your answer.

a) What is the dimension of Row A?

b) What is the dimension of (Row $A)^\perp$?

Solution.

a) It is standard fact that the dimension of the row space of A is the same as the dimension of the column space of A, which is 3.

b) By 6.2 facts and the Rank Theorem: $\dim((\operatorname{Row} A)^{\perp}) = \dim(\operatorname{Nul} A) = 8-3 = 5$.

Alternatively, since Row A has dimension 3 and is a subspace of $\mathbf{R}^8$, its orthogonal complement is a 5-dimensional subspace of $\mathbf{R}^8$ just by properties of orthogonal complements.