Math 1553 Worksheet §5.6 - §6.5

Solutions

1. Courage Soda and Dexter Soda compete for a market of 210 customers who drink soda each day.

Today, Courage has 80 customers and Dexter has 130 customers. Each day:

70% of Courage Soda's customers keep drinking Courage Soda, while 30% switch to Dexter Soda.

40% of Dexter Soda's customers keep drinking Dexter Soda, while 60% switch to Courage Soda.

a) Write a stochastic matrix A and a vector x so that Ax will give the number of customers for Courage Soda and Dexter Soda (in that order) tomorrow.
You do not need to compute Ax.

$$A = \begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{pmatrix} \text{ and } x = \begin{pmatrix} 80 \\ 130 \end{pmatrix}.$$

b) Find the steady-state vector for A.

$$\left(A - I \mid 0\right) = \left(\begin{array}{cc|c} -0.3 & 0.6 & 0 \\ 0.3 & -0.6 & 0 \end{array}\right) \xrightarrow[R_1=R_1/(-0.3)]{R_2=R_2+R_1} \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

so $x_1 = 2x_2$ and x_2 is free. A 1-eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, so the steady state vector is $w = \frac{1}{2+1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$.

c) Use your answer from (b) to determine the following: in the long run, roughly how many daily customers will Courage Soda have?

As n gets large, $A^n \begin{pmatrix} 80 \\ 130 \end{pmatrix}$ approaches $210 \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 140 \\ 70 \end{pmatrix}$. Courage will have roughly 140 customers.

2. True/False

(1) If u is in subspace W, and u is also in $W^{\perp}$, then $u = 0$.

(2) If y is in a subspace W, the orthogonal projection of y onto $W^{\perp}$ is 0.

(3) If x is orthogonal to v and w, then x is also orthogonal to $v - w$.

Solution.

(1) TRUE: Such a vector u would be orthogonal to itself, so $u \cdot u = ||u||^2 = 0$. Therefore, u has length 0, so $u = 0$.

(2) TRUE: y is in W, so $y \perp W^{\perp}$. Its orthogonal projection onto W is y and orthogonal projection onto $W^{\perp}$ is 0. In fact y has orthogonal decomposition $y = y + 0$, where y is in W and 0 is in $W^{\perp}$.

(3) TRUE: By properties of the dot product, if x is orthogonal to v and w then x is orthogonal to everything in Span$\{v,w\}$ (which includes $v-w$).

3. **a)** Find the standard matrix B for proj_L, where $L = \text{Span}\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\}$.

b) What are the eigenvalues of B? Is B is diagonalizable?

Solution.

a) We use the formula $B = \frac{1}{u\cdot u}uu^T$ where $u = \begin{pmatrix}1\\1\\-1\end{pmatrix}$ (this is the formula $B = A(A^TA)^{-1}A^T$ when "A" is just the single vector u).

$$B = \frac{1}{1(1)+1(1)+(-1)(-1)}\begin{pmatrix}1\\1\\-1\end{pmatrix}\begin{pmatrix}1 & 1 & -1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & 1 & -1\\1 & 1 & -1\\-1 & -1 & 1\end{pmatrix}$$

$$\implies B = \frac{1}{3}\begin{pmatrix}1 & 1 & -1\\1 & 1 & -1\\-1 & -1 & 1\end{pmatrix}.$$

b) $Bx = x$ for every x in L, and $Bx = 0$ for every x in $L^\perp$, so B has two eigenvalues: $\lambda_1 = 1$ with algebraic and geometric) multiplicity 1, $\lambda_2 = 0$ with algebraic and geometric multiplicity 2. Since the algebraic and geometric multiplicities are the same for each eigenvalue, we know B is diagonalizable. We can actually compute the diagonalization of B (we're not asked in the question). Here $v_1 = \begin{pmatrix}1\\1\\-1\end{pmatrix}$ is an eigenvector for $\lambda_1 = 1$, whereas $v_2 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$ and $v_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}$ are eigenvectors for $\lambda_2 = 0$. Therefore

$$B = \begin{pmatrix}1 & 1 & 1\\1 & -1 & 0\\-1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\1 & -1 & 0\\-1 & 0 & 1\end{pmatrix}^{-1}$$

4. $y = \begin{pmatrix}0\\2\\4\end{pmatrix}$, $u_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $u_2 = \begin{pmatrix}-1\\1\\0\end{pmatrix}$

(1) Determine whether u_1 and u_2
   (a) are linearly independent
   (b) are orthogonal
   (c) span $\mathbf{R}^3$
(2) Is y in $W = \text{Span}\{u_1, u_2\}$?
(3) Compute the vector w that most closely approximates y within W.
(4) Construct a vector, z, that is in $W^\perp$.
(5) Make a rough sketch of W, y, w, and z.

Solution.

(1) A quick check shows that the vectors u_1 and u_2 are orthogonal and linearly independent, so Span$\{u_1, u_2\}$ is a plane in $\mathbf{R}^3$, but is not all of $\mathbf{R}^3$.

(2) By inspection, y is not in the span because it has a non-zero x_3 component.

(3) The vector w is $\text{proj}_W y$. The orthogonal projection of y onto W is calculated in the usual way.

$$A^T Av = A^T b$$

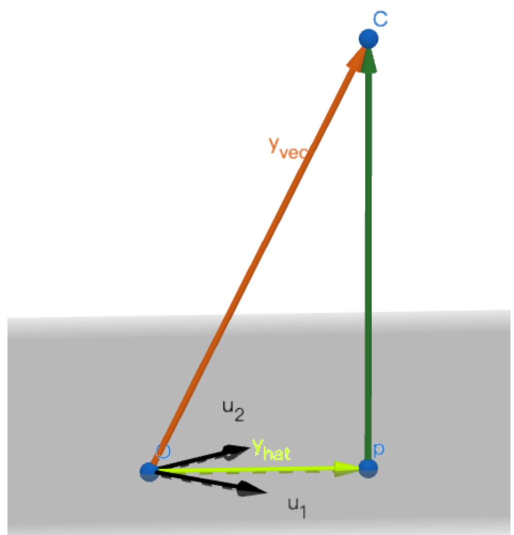
$$A^T A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \text{so} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} v = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$w = Av = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

Another quick way to do this problem is note that W is the xy-plane of $\mathbf{R}^3$, so the projection of $\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$ onto W is just $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

(4) One vector in $W^\perp$ is $z = y - \text{proj}_W y = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$.

(5) Here is a picture. The vector w is labeled "y_{hat}" in the drawing.



5. Find the best fit line $y = Ax + B$ through the points $(0,0)$, $(1,8)$, $(3,8)$, and $(4,20)$.

Solution.

We want to find a least squares solution to the system of linear equations

$$\begin{aligned} 0 &= A(0) + B \\ 8 &= A(1) + B \\ 8 &= A(3) + B \\ 20 &= A(4) + B \end{aligned} \qquad \Longleftrightarrow \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

We compute

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 26 & 8 \\ 8 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 112 \\ 36 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 26 & 8 & 112 \\ 8 & 4 & 36 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right).$$

Hence the least squares solution is $A = 4$ and $B = 1$, so the best fit line is $y = 4x + 1$.