Math 1553 Worksheet §2.3, S2.4

Solutions

1. True or false. If the statement is *always* true, answer True. Otherwise, answer False. In parts (a) and (b), A is an $m \times n$ matrix and b is a vector in $\mathbf{R}^m$.

a) If b is in the span of the columns of A, then the matrix equation $Ax = b$ is consistent.

b) If $Ax = b$ is inconsistent, then A does not have a pivot in every column.

c) If A is a 4×3 matrix, then the equation $Ax = b$ is inconsistent for some b in $\mathbb{R}^4$.

Solution.

a) True. Let the columns of A be $v_1, \cdots, v_n$. Since b in Span$\{v_1, \cdots, v_n\}$, this means b can be written as a linear combinations of these column vectors, so

$$x_1 v_1 + \cdots + x_n v_n = b$$

for some scalars $x_1, \ldots, x_n$. Therefore, $Ax = b$ where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

b) False, for instance consider

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

This is an inconsistent system even though A has a pivot in each column.

c) True. Any 4×3 matrix A will have at most 3 pivots, so A cannot have a pivot in every row. For example, consider the augmented matrix $\left(A \mid b\right)$ below.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

2. Let

$$A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}.$$

Solve the matrix equation $Ax = b$ and write your answer in parametric form.

Solution.

We translate the matrix equation into an augmented matrix, and row reduce it:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The right column is not a pivot column, so the system is consistent.

The RREF of the augmented matrix gives

$$x_1 = 2 - 5x_3 \qquad x_2 = 3 - 4x_3 \qquad x_3 = x_3 \;\; (x_3 \text{ is free}).$$

If we wanted to write just one specific solution, we could take $x_3 = 0$ and that would give us $x_1 = 2, x_2 = 3$:

$$b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}.$$

3. Find the set of solutions to $x_1 - 3x_2 + 5x_3 = 0$. Next, find the set of solutions to $x_1 - 3x_2 + 5x_3 = 3$. In each case, write your solution in parametric vector form. How do the solution sets compare geometrically?

Solution.

The homogeneous system $x_1 - 3x_2 + 5x_3 = 0$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 0 \end{array}\right)$, which has two free variables x_2 and x_3.

$$x_1 = 3x_2 - 5x_3 \qquad x_2 = x_2 \text{ (free)} \qquad x_3 = x_3 \text{ (free)}.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

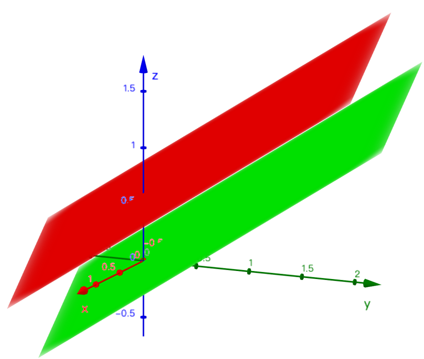
The solution set for $x_1 - 3x_2 + 5x_3 = 0$ is the plane spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.

The nonhomogeneous system $x_1 - 3x_2 + 5x_3 = 3$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 3 \end{array}\right)$ which has two free variables x_2 and x_3.

$$x_1 = 3 + 3x_2 - 5x_3 \qquad x_2 = x_2 \qquad x_3 = x_3.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 + 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

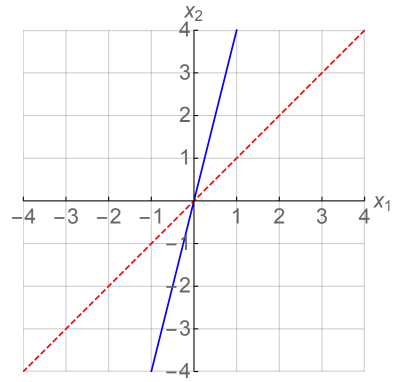
This solution set (red) is the *translation by* $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ of the plane (green) spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.



Here is the link to a 3D picture you can play with https://www.geogebra.org/calculator/j57ttsnb

4. Let $A = \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}$. Draw the span of the columns of A, and draw the set of solutions to $Ax = 0$. Clearly label each.

Solution.



The blue line is the span of columns of A: Span $\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right\}$. If you draw the two column vectors, you will see they both fall on the line $x_2 = 4x_1$.

The red dashed line is the span of solutions of $Ax = 0$: Span $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$. To see this is the case, you can row reduce the augmented matrix to RREF, which is $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$. That implies the solution set is the line $x_2 = x_1$.

5. Write an augmented matrix corresponding to a system of two linear equations in the three variables x_1, x_2, x_3, so that the solution set is the span of $\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$.

Solution.

We are asked to come up with a system whose solution set is the prescribed span, rather than being handed a system and discovering its solution set.

Since the span of any vector includes the origin, the zero vector is a solution, so the system is homogeneous.

Note that the span of $\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$ is all vectors of the form $t\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$ where t is real.

It consists of all $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ so that $x_1 = -4x_2,\ \ x_2 = x_2,\ \ x_3 = 0$.

The equation $x_1 = -4x_2$ gives $x_1 + 4x_2 = 0$, so one line in the matrix can be $\left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \end{array}\right)$.

The equation $x_3 = 0$ translates to $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \end{array}\right)$. Note that this leaves x_2 free, as desired.

This gives us the augmented matrix

$$\boxed{\left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)}.$$

(Multiple examples are possible. For example do an arbitrary row operation on the above matrix, that will also work.)