Math 1553 Worksheet §4.1 - §5.1

Solutions

1. Let $A = \begin{pmatrix} 7 & 1 & 4 & 1 \\ -1 & 0 & 0 & 6 \\ 9 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

a) Compute $\det(A)$.

b) Compute $\det(A^{-1})$ without doing any more work.

c) Compute $\det((A^T)^5)$ without doing any more work.

d) Find the volume of the parallelepiped formed by the columns of A.

Solution.

a) The second column has three zeros, so we expand by cofactors:

$$\det\begin{pmatrix} 7 & 1 & 4 & 1 \\ -1 & 0 & 0 & 6 \\ 9 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\det\begin{pmatrix} -1 & 0 & 6 \\ 9 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

Now we expand the second column by cofactors again:

$$\cdots = -2\det\begin{pmatrix} -1 & 6 \\ 0 & -1 \end{pmatrix} = (-2)(-1)(-1) = -2.$$

b) From our notes, we know $\det(A^{-1}) = \dfrac{1}{\det(A)} = -\dfrac{1}{2}$.

c) $\det(A^T) = \det(A) = -2$. By the multiplicative property of determinants, if B is any $n \times n$ matrix, then $\det(B^n) = (\det B)^n$, so

$$\det((A^T)^5) = (\det A^T)^5 = (-2)^5 = -32.$$

d) Volume of the parallelepiped is $|\det(A)| = 2$

2. Let A be an $n \times n$ matrix.

a) If $\det(A) = 1$ and c is a scalar, what is $\det(cA)$?

b) Using cofactor expansion, explain why $\det(A) = 0$ if A has adjacent identical columns.

Solution.

a) By the properties of the determinant, scaling one row by c multiplies the determinant by c. When we take cA for an $n \times n$ matrix A, we are multiplying *each* row by c. This multiplies the determinant by c a total of n times. Thus, if A is $n \times n$ and $\det(A) = 1$, then

$$\det(cA) = c^n \det(A) = c^n(1) = c^n.$$

b) If A has identical adjacent columns, then the cofactor expansions will be identical, except the signs of the cofactors will be opposite (due to the $(-1)^{\text{power}}$ factors).

Therefore, $\det(A) = -\det(A)$, so $\det A = 0$.

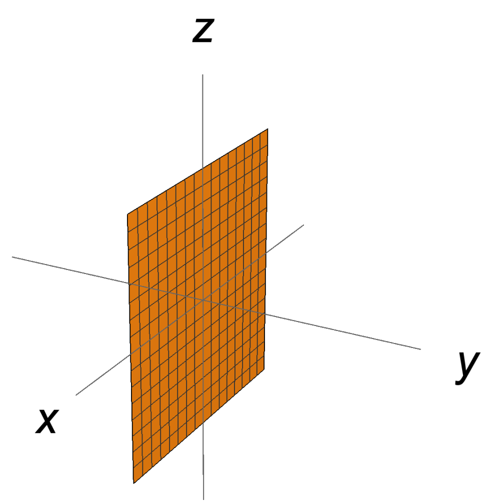
3. In what follows, T is a linear transformation with matrix A. Find the eigenvectors and eigenvalues of A without doing any matrix calculations. (Draw a picture!)

a) $T : \mathbf{R}^3 \to \mathbf{R}^3$ that projects vectors onto the xz-plane in $\mathbf{R}^3$.

b) $T : \mathbf{R}^2 \to \mathbf{R}^2$ that reflects vectors over the line $y = 2x$ in $\mathbf{R}^2$.

Solution.

a) Here is a picture:

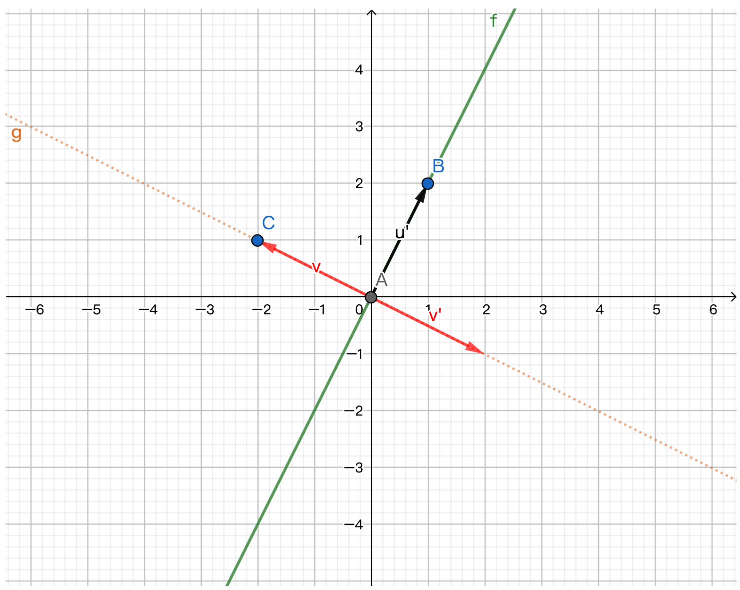


$T(x, y, z) = (x, 0, z)$, so T fixes every vector in the xz-plane and destroys every vector of the form $(0, a, 0)$ with a real. Therefore, $\lambda = 1$ and $\lambda = 0$ are eigenvalues and in fact they are the only eigenvalues since their combined eigenvectors span all of $\mathbf{R}^3$.

The eigenvectors for $\lambda = 1$ are all vectors of the form $\begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$ where at least one of x and z is nonzero, and the eigenvectors for $\lambda = 0$ are all vectors of the form $\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$ where $y \neq 0$. In other words:

The 1-eigenspace consists of all vectors in Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, while the 0-eigenspace consists of all vectors in Span $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

b) Here is the picture you can play with https://www.geogebra.org/calculator/xxmhzgev



T fixes every vector along the line $y = 2x$, so $\lambda = 1$ is an eigenvalue and its eigenvectors are all vectors $\begin{pmatrix} t \\ 2t \end{pmatrix}$ where $t \neq 0$.
T flips every vector along the line perpendicular to $y = 2x$, which is $y = -\frac{1}{2}x$ (for example, $T(-2,1) = (2,-1)$). Therefore, $\lambda = -1$ is an eigenvalue and its eigenvectors are all vectors of the form $\begin{pmatrix} s \\ -\frac{1}{2}s \end{pmatrix}$ where $s \neq 0$.

4. True or false. If the statement is always true, answer true and justify why it is true. Otherwise, answer false and give an example that shows it is false. In every case, assume that A is an $n \times n$ matrix.

a) The entries on the main diagonal of A are the eigenvalues of A.

b) The number λ is an eigenvalue of A if and only if there is a nonzero solution to the equation $(A - \lambda I)x = 0$.

c) To find the eigenvectors of A, we reduce the matrix A to row echelon form.

d) If A is invertible and 2 is an eigenvalue of A, then $\frac{1}{2}$ is an eigenvalue of A^{-1}.

e) If $\mathrm{Nul}(A)$ has dimension at least 1, then 0 is an eigenvalue of A and $\mathrm{Nul}(A)$ is the 0-eigenspace of A.

Solution.

a) False. This is true if A is triangular, but not in general.

For example, if $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ then the diagonal entries are 2 and 0 but the only eigenvalue is $\lambda = 1$, since solving the characteristic equation gives us

$$(2-\lambda)(-\lambda)-(1)(-1)=0 \qquad \lambda^2-2\lambda+1=0 \qquad (\lambda-1)^2=0 \qquad \lambda=1.$$

b) True.

$$(A-\lambda I)x = 0 \iff Ax - \lambda x = 0 \iff Ax = \lambda x.$$

Therefore, $(A-\lambda I)x = 0$ has a nonzero solution if and only if $Ax = \lambda x$ has a nonzero solution, which is to say that λ is an eigenvalue of A.

c) False. Eigenvalues come from the equation $(A-\lambda I)x = 0$, so if we want to find eigenvalues with row reduction, we would need to row-reduce $\left(A-\lambda I \mid 0\right)$ (rather than $\left(A \mid 0\right)$) to determine the values for λ so that $A-\lambda I$ is non-invertible. Generally, instead of doing this we will set $\det(A-\lambda I) = 0$ in section 5.2.

d) True. Let v be an eigenvector corresponding to the eigenvalue 2.

$$Av = 2v \implies A^{-1}Av = A^{-1}(2v) \implies v = 2A^{-1}v \implies \frac{1}{2}v = A^{-1}v.$$

Therefore, v is an eigenvector of A^{-1} corresponding to the eigenvalue $\frac{1}{2}$.

e) True. For every v in Nul A, we have $Av = 0v$. If $v \neq 0$, this is exactly the definition of v being an eigenvector corresponding to the eigenvalue 0. If Nul$A$ has dimension at least 1, then infinitely many nonzero vectors satisfy $Av = 0$, so 0 is an eigenvalue of A (and every nonzero vector v satisfying $Av = 0$ is an eigenvector of A) and Nul A is the 0-eigenspace of A.