MATH 1553
SAMPLE FINAL EXAM, SPRING 2022

Name		**GT ID**	

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 100 points, and you have 170 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

> This is a practice exam. It is meant to be roughly similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam.

____

Problem 1.

[1 point each]

TRUE or FALSE. Circle **T** if the statement is *always* true. Otherwise, answer **F**. You do not need to show work or justify your answer.

a) **T** **F** If $T : \mathbf{R}^3 \to \mathbf{R}^3$ is a linear transformation that satisfies

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}, \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\0\\0\end{pmatrix},$$

then T is one to one.

b) **T** **F** If the system $Ax = \begin{pmatrix}1\\-2\end{pmatrix}$ has a unique solution $\begin{pmatrix}3\\1\end{pmatrix}$, then the homogeneous equation $Ax = 0$ has only the trivial solution.

c) **T** **F** If A and B are $n \times n$ matrices then $\det(A+B) = \det(A) + \det(B)$.

d) **T** **F** If A is a 5×7 and $\dim(\text{Nul } A) = 4$, then $\dim(\text{Row } A) = 3$.

e) **T** **F** The set $W = \left\{ \begin{pmatrix}x\\y\\z\\w\end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - y = z - w \right\}$ is a 3-dimensional subspace of $\mathbf{R}^4$.

f) **T** **F** If A is a 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda(2-\lambda)(3-\lambda),$$

then A is diagonalizable.

g) **T** **F** Suppose W is a subspace of $\mathbf{R}^n$ and B is the matrix for orthogonal projection onto W. Then for every x in $\mathbf{R}^n$, we have $Bx = x$ or $Bx = 0$.

h) **T** **F** Each inconsistent system $Ax = b$ has exactly one least squares solution.

i) **T** **F** Any $n \times n$ matrix with n linearly independent eigenvectors in $\mathbf{R}^n$ is diagonalizable.

j) **T** **F** The vector $\begin{pmatrix}0.6\\0.4\end{pmatrix}$ is the steady state vector of the matrix $\begin{pmatrix}0.4 & 0.6\\0.6 & 0.4\end{pmatrix}$.

Problem 2.

Short answer. You do not need to show your work on (a) or (b), but briefly show your work on part (c).

a) Let $\{v_1, v_2, \dots, v_n\}$ be vectors in $\mathbf{R}^m$. Which of the following conditions imply that these vectors are linearly independent? Circle all that apply.

(i) The vector equation $x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0$ has a unique solution.

(ii) The subspace Span$\{v_1, v_2, \dots, v_n\}$ has dimension n.

(iii) The RREF of the matrix $A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix}$ has a pivot in every column.

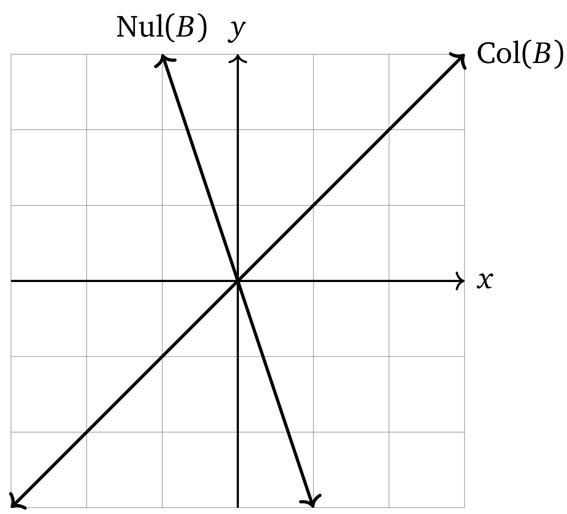
b) Let A be a 3×4 matrix. Which of the following statements *can* be true? Circle all that apply.

(i) The transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ defined by $T(x) = Ax$ is one to one.

(ii) The rank of A is equal to 2 and Nul(A) is the x-axis.

(iii) The column space of A and the null space of A have the same dimension.

c) The null space and column space of another matrix B are given in the picture. Write such a matrix B.



Problem 3.

Short answer. You do not need to show your work.

a) Suppose A is an $m \times n$ matrix and the only solution to the homogeneous equation $Ax = 0$ is the trivial solution $x = 0$. Let T be the matrix transformation $T(x) = Ax$. Which of the following *must* be true? Circle all that apply.

(i) T is onto.

(ii) T is one-to-one.

(iii) If $m = n$, then A is invertible.

(iv) If $m > n$, then the equation $Ax = b$ is inconsistent for at least one b in $\mathbf{R}^m$.

b) The equation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 3 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ has least-squares solution $\widehat{x} = \begin{pmatrix} 2/5 \\ -2 \end{pmatrix}$. What is the closest vector to $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ in Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\}$? Enter your answer here: $\begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$

c) Let W be the set of all vectors in $\mathbf{R}^3$ of the form (a, b, a) where a and b are real numbers. Which of the following is $W^\perp$?

(i) Span $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

(ii) Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

(iii) Nul$\begin{pmatrix} a & b & a \end{pmatrix}$

(iv) Nul$\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$

(v) none of these

Problem 4.

Short answer. Assume that the entries in all matrices are real numbers. You do not need to show your work or justify your answers.

a) Suppose A is a 2×2 matrix with eigenvalues $\lambda = 1$ and $\lambda = 3$. What is the characteristic polynomial of A^2?

b) Give an example of a 3×3 matrix with eigenvector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

c) Give an example of a 2×2 matrix that has no real eigenvalues.

d) Give an example of a 3×3 matrix A with exactly one eigenvalue $\lambda = 2$, so that the 2-eigenspace of A is a line.

Problem 5.

You do not need to show your work, and there is no partial credit except in part (d).

a) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear transformation and suppose $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ are in the range of T. Write another nonzero vector in the range of T here: $\begin{pmatrix} \\ \\ \end{pmatrix}$

b) Suppose that A is a 12×9 matrix and the solution set to $Ax = 0$ has dimension 7.

(i) Fill in the blank: the dimension of the column space of A is ________.

(ii) Fill in the blank: the dimension of the row space of A is ________.

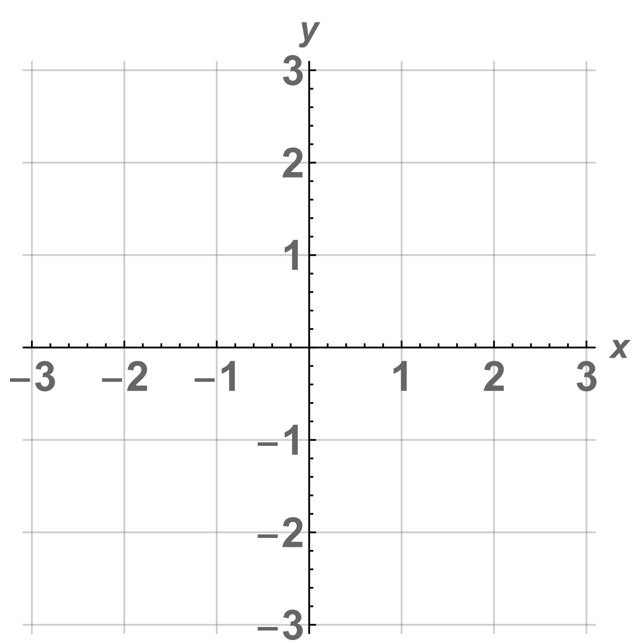
c) Suppose A is a stochastic matrix. Which of the following must be true? Circle all that apply.

(i) The sum of entries in each row of A is equal to 1.

(ii) The sum of entries in each column of A is equal to 1.

(iii) No entry of A is greater than 1.

d) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation of reflection across the line $y = 3x$, and let A be the standard matrix for T. Draw each eigenspace of A precisely, and clearly label each eigenspace with its eigenvalue.



Problem 6.

Short answer. On parts (a), (b), and (c), you do not need to show your work and there is no partial credit. Briefly show your work in part (d).

a) Suppose that v and w are eigenvectors of a matrix A corresponding to the eigenvalues 4 and -1, respectively. Find $A(2v+3w)$ in terms of v and w.

b) Suppose that for two 5×5 matrices A and B, we have

$$\det(A) = 3, \quad \det(A^{-1}B) = 7.$$

Find $\det(B)$.

c) Suppose that A is a positive stochastic matrix with steady-state vector $\begin{pmatrix} 3/10 \\ 7/10 \end{pmatrix}$. What vector does $A^n \begin{pmatrix} 350 \\ 50 \end{pmatrix}$ approach as n becomes very large?

d) Compute the orthogonal projection of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ onto $\operatorname{Span}\left\{\begin{pmatrix} 2 \\ -3 \end{pmatrix}\right\}$.

Problem 7.

a) Find the matrix of the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ which rotates by 90° counterclockwise. Enter your answer here: $\begin{pmatrix} & \\ & \end{pmatrix}$

b) Find the matrix of the transformation defined by $U\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ 3y - x \\ x - 3y \end{pmatrix}$.

Enter your answer here: $\begin{pmatrix} & \\ & \end{pmatrix}$

c) Circle which transformation makes sense: $T \circ U$ $\quad U \circ T$

Find the standard matrix for the transformation you circled, and enter it below.

$$\begin{pmatrix} & \\ & \end{pmatrix}$$

d) Find A^{-1} if $A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$.

Problem 8.

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 4 \\ 1 & 1 & 1 \end{pmatrix}$$

a) Find the characteristic polynomial and the eigenvalues of A.

b) For each eigenvalue of A, find one corresponding eigenvector.

c) Find an invertible 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$.

Problem 9.

Let $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 + x_2 + x_3 = 0 \right\}$ and $x = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

(i) Find a basis for W.

(ii) Find x_W, the orthogonal projection of x onto W.

(iii) Find $x_{W^\perp}$

Problem 10.

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0,0), \qquad (1,2), \qquad (3,-1).$$

Enter your answer below:

$$y = \_\_\_\_\_\_ x \; + \; \_\_\_\_\_\_ .$$

You must show appropriate work. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.