MATH 1553, SPRING 2022
FINAL EXAM

Name		GT ID	

Circle your lecture below.

Jankowski, lecture A (8:25-9:15 AM) Jankowski, lecture D (9:30-10:20 AM)

Yu, lecture G (12:30-1:20 PM)

Leykin, lecture I (2:00-2:50 PM) Leykin, lecture M (3:30-4:20 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 100 points, and you have 170 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- Simplify your answers as much as possible. For example, you may lose points if you do not simplify $\frac{8}{2}$ to 4, or if you do not simplify $\frac{0.1}{0.9}$ to $\frac{1}{9}$, etc.
- As always, RREF means "reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Unless specified otherwise, the entries of all matrices are real numbers.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 9:00 PM on Tuesday, May 3.

____

Problem 1.

True or false. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 1 point.

a) Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x+y+z=1 \right\}$. Then W is a subspace of $\mathbf{R}^3$.

TRUE FALSE

b) If A is a matrix with more rows than columns, then the transformation $T(x) = Ax$ cannot be one-to-one.

TRUE FALSE

c) If A is a 5×11 matrix, then $\dim(\mathrm{Nul}\ A)$ must be greater than $\dim(\mathrm{Col}\ A)$.

TRUE FALSE

d) If A is a 3×3 matrix with characteristic polynomial $\det(A - \lambda I) = -(\lambda+1)^2(\lambda-2)$, then the 2-eigenspace of A is a line.

TRUE FALSE

e) If A is an $n \times n$ matrix and λ is an eigenvalue of A, then the columns of $A - \lambda I$ are linearly dependent.

TRUE FALSE

f) Let A be a 3×3 matrix. If $\dim(\mathrm{Nul}\ A) = 2$, then all of the eigenvalues of A must be real.

TRUE FALSE

g) Every inconsistent linear system of equations has exactly one least-squares solution.

TRUE FALSE

h) Let $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. Then $W^\perp = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

TRUE FALSE

i) If A is a 5×5 matrix, then $(\mathrm{Row}\ A)^\perp = \mathrm{Nul}\ A$.

TRUE FALSE

j) Let W be a subspace of $\mathbf{R}^n$ and v be a vector in $\mathbf{R}^n$. If the orthogonal projection of v onto W is the zero vector, then v must be in $W^\perp$.

TRUE FALSE

Solution.

a) False. W does not contain the zero vector, since $0+0+0\neq 1$.

b) False. For example, if $A=\begin{pmatrix}1&2\\1&3\\1&4\end{pmatrix}x$ then $T(x)=Ax$ is one-to-one.

c) True. By the Rank Theorem, $\dim(\text{Nul }A)+\dim(\text{Col }A)=11$ and $\dim(\text{Col }A)\leq 5$ since Col A is a subspace of $\mathbf{R}^5$, thus $\dim(\text{Nul}A)\geq 6$.

Alternative reasoning: since A is 5×11, it has at most 5 pivots (so $\text{rank}(A)\leq 5$), so the homogeneous equation $Ax=0$ has at least 6 free variables (thus $\text{nullity}(A)\geq 6$).

d) True. The algebraic multiplicity of $\lambda=2$ is one. Since (alg. mult) $\geq$ (geo. mult.) ≥ 1, this means the geometric multiplicity of $\lambda=2$ is one, so the 2-eigenspace is a line.

e) True: From the fact that λ is an eigenvalue of A, we know that $A-\lambda I$ is not invertible, so $A-\lambda I$ has fewer than n pivots and therefore its columns are linearly dependent.

f) True: A is 3×3 and its 0-eigenspace is 2-dimensional, so the algebraic multiplicity of $\lambda=0$ is at least 2. If A had an eigenvalue λ that was not real, then $\overline{\lambda}$ would also be an eigenvalue. This is impossible, since this would mean the degree 3 polynomial would have at least 4 roots counting algebraic multiplicities.

g) False: the columns of the corresponding matrix A might be linearly dependent. For example,

$$\begin{pmatrix}1&1\\1&1\end{pmatrix}x=\begin{pmatrix}0\\1\end{pmatrix}$$

will have infinitely many least-squares solutions.

h) True. The vector $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ is orthogonal to both $\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$ since $\begin{pmatrix}-1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=0$ and $\begin{pmatrix}-1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}=0$, so $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ is in $W^\perp$. Here $\dim(W^\perp)=1$ since W is a 2-dimensional subspace of $\mathbf{R}^3$, so $\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$ is a basis for $W^\perp$.

i) True. This is a fact from section 6.2 for any $n\times n$ matrix A.

j) True: by orthogonal decomposition we have $v=v_W+v_{W^\perp}$ and we have been told $v_W=0$, so $v=v_{W^\perp}$. In other words v is in $W^\perp$.

Problem 2.

Multiple choice and short answer. You do not need to show your work, and there is no partial credit.

a) (3 points) Suppose v_1, v_2, v_3 are vectors in $\mathbf{R}^5$. Which of the following statements guarantee that $\{v_1, v_2, v_3\}$ is linearly independent? Circle *all* that apply.

(i) v_3 is not a linear combination of v_1 and v_2.

(ii) $\dim\big(\text{Span}\{v_1, v_2, v_3\}\big) = 3$.

(iii) For some b in $\mathbf{R}^5$, the equation $x_1v_1 + x_2v_2 + x_3v_3 = b$ has exactly one solution.

b) (3 points) Let A and B be invertible $n \times n$ matrices. Which of the following matrices must also be invertible? Circle *all* that apply.

(i) AB

(ii) $A + B$

(iii) $-3A$

c) (4 points) Let $V = \text{Nul}\begin{pmatrix} 1 & 4 & 4 \end{pmatrix}$. Which of the following options are a basis of V? Circle *all* that apply.

(i) $\left\{\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}\right\}$

(ii) $\left\{\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}\right\}$

(iii) $\left\{\begin{pmatrix} 8 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}\right\}$

(iv) $\left\{\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}\right\}$

Solution.

a) (ii) and (iii) are true. Part (i) is false, and one example where (i) holds but $\{v_1, v_2, v_3\}$ is linearly dependent is $v_1 = e_1$, $v_2 = 2e_1$, $v_3 = e_2$. Part (ii) is true by the increasing span criterion. For part (iii) we know that if $x_1v_1 + x_2v_2 + x_3v_3 = b$ has a unique solution for some b then $x_1v_1 + x_2v_2 + x_3v_3 = 0$ has exactly one solution (namely the trivial solution) which is the very definition of linear independence.

b) (i) and (iii) are true. For part (i) we know $(AB)^{-1} = B^{-1}A^{-1}$. Part (ii) is false, for example $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are both invertible but $A + B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ which is not invertible. Part (iii) is true and can be seen in a variety of ways, for example since A is invertible we have $\det(A) \neq 0$ so $\det(-3A) = (-3)^n \det(A) \neq 0$.

c) (ii) and (iii). Since $\begin{pmatrix} 1 & 4 & 4 \end{pmatrix}$ has one pivot, its homogeneous solution set will have two free variables, so $\dim(V) = 2$ and therefore (i) and (iv) are incorrect.

We can check that (ii) and (iii) are each a basis for V just by computation: in each case, the two vectors are linearly independent and $\begin{pmatrix} 1 & 4 & 4 \end{pmatrix}$ times each of the vectors is zero, so they form a basis for V.

Alternative method: we could compute a basis for V by putting $\left(\begin{array}{ccc|c} 1 & 4 & 4 & 0 \end{array}\right)$ in parametric vector form and getting $x_1 + 4x_2 + 4x_3 = 0$ where x_2 and x_3 are free. This gives

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4x_2 - 4x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, (ii) is a basis for V. For (iii), call $v_1 = \begin{pmatrix} 8 \\ -1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$. Note that $v_1 = -\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = 2\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$, so $\{v_1, v_2\}$ is a set of 2 linearly independent vectors in the two-dimensional subspace V and is therefore (iii) a basis for V by the Basis Theorem.

Problem 3.

Multiple choice and short answer. You do not need to show your work, and there is no partial credit.

a) (2 pts) Suppose A is a 40×50 matrix and the dimension of the null space of A is 12. Which *one* of the following describes the column space of A?

(i) Col(A) is a 38-dimensional subspace of $\mathbf{R}^{50}$.

(ii) Col(A) is a 28-dimensional subspace of $\mathbf{R}^{50}$.

(iii) Col(A) is a 38-dimensional subspace of $\mathbf{R}^{40}$.

(iv) Col(A) is a 28-dimensional subspace of $\mathbf{R}^{40}$.

b) (4 points) Let $A = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}$.

(i) Find A^{-1}. Clearly circle your answer below.

(I) $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$ (II) $A^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix}$ (III) $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix}$

(IV) $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}$ (V) $A^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}$ (VI) $A^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 2 \\ 1 & -4 \end{pmatrix}$

(ii) Find det($3A$).

(I) 2 (II) 6 (III) 9 (IV) 18 (V) 36 (VI) none of these

c) (4 points) Let A be an $n \times n$ matrix and let T be the matrix transformation $T(x) = Ax$. Which of the following conditions *guarantee* that A is invertible? Select *all* that apply.

(i) For some b in $\mathbf{R}^n$, the equation $Ax = b$ has exactly one solution.

(ii) The matrix transformation $T(x) = Ax$ is onto.

(iii) A has exactly one eigenvalue $\lambda = 2$.

(iv) For each x in $\mathbf{R}^n$, there is a vector y in $\mathbf{R}^n$ so that $T(x) = y$.

Solution.

a) The answer is (iii): Col A is a subspace of $\mathbf{R}^{40}$, and by the Rank Theorem,

$$\dim(\text{Col } A) + \dim(\text{Nul } A) = 50, \quad \text{thus} \quad \dim(\text{Col } A) = 50 - 12 = 38.$$

b) For part (i), the answer is (III).

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix}.$$

Alternatively, the student could just multiply each of the candidates for A^{-1} by A to find that (III) is the inverse of A.

For part (ii), the answer is (IV). We could either compute $3A = \begin{pmatrix} 12 & 3 \\ 6 & 3 \end{pmatrix}$ and find its determinant is 18 directly, or we could use the fact that $\det(3A) = 3^2 \det(A) = 9(4(1)-2(1)) = 9(2) = 18$.

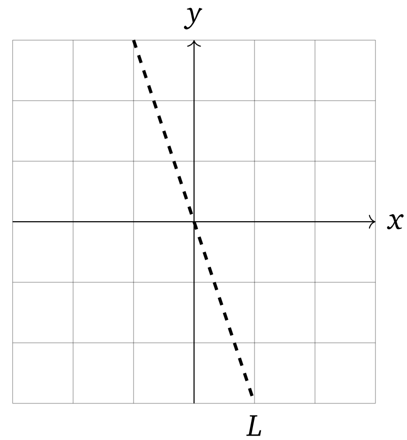
c) (i), (ii), and (iii) guarantee that A is invertible by the Invertible Matrix Theorem.

Part (i) implies that $Ax = 0$ has only the trivial solution, part (ii) is part of the IMT, and part (iii) implies that 0 is not an eigenvalue of A. Part (iv) is just part of the definition of transformation, so it does not guarantee that A is invertible, for example $T(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x$ satifies (iv) but A is not invertible.

Problem 4.

Short answer and multiple choice. You do not need to show your work on this problem.

a) (4 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across the **dashed** line L below, and let A be the standard matrix for T.



(i) Write all eigenvalues of A. ____________

(ii) For each eigenvalue of A, draw one eigenvector on the graph above. Clearly label the eigenvalue that corresponds to each eigenvector.

b) (4 points) Let A be a 4×3 matrix such that $\text{rank}(A) = 1$. Let $T(v) = Av$ be the associated matrix transformation. For each question below, circle one answer.

(i) What is the domain of T? $\mathbf{R}$ $\mathbf{R}^2$ a line in $\mathbf{R}^3$ $\mathbf{R}^3$ $\mathbf{R}^4$

(ii) What is the codomain of T? $\mathbf{R}$ $\mathbf{R}^2$ a line in $\mathbf{R}^3$ $\mathbf{R}^3$ $\mathbf{R}^4$

(iii) What is the range of T, geometrically?
a point a line a plane 3-dimensional space

(iv) Fill in the blank: $\dim(\text{Row } A) =$ ____________.

c) (2 points) Suppose A is a 2×2 positive stochastic matrix with the property that as n gets very large, $A^n \begin{pmatrix} 200 \\ 300 \end{pmatrix}$ approaches $\begin{pmatrix} 100 \\ 400 \end{pmatrix}$. What is the steady-state vector w for A? Write w in the space below, and *simplify your answer*.

$$w = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

Solution.

a) We have done this problem many times: it is in our PDF notes, in the 5.1 worksheet, in the 5.1 supplement, in 3d from sample midterm 3, and 5d from the practice final, at least!

For part (i), the eigenvalues are 1 and -1. We have seen this so often that there is just no excuse for missing either of these eigenvalues.

For part (ii), any nonzero vector along the dotted line (i.e. in the span of $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvector for $\lambda = 1$, while any nonzero vector perpendicular to the dotted line (i.e. in the span of $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$) is in the (-1)-eigenspace of A.

b) Part (i), the domain of T is $\mathbf{R}^3$.
Part (ii): the codomain of T is $\mathbf{R}^4$.
Part (iii): the range of T is a line (in $\mathbf{R}^4$) since the rank of A is 1.
Part (iv): $\dim(\mathrm{Row}\ A) = \dim(\mathrm{Col}\ A) = 1$.

c) $A^n \to 500\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 100 \\ 400 \end{pmatrix}$, thus

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{100}{500} \\ \\ \frac{400}{500} \end{pmatrix} = \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix}.$$

Problem 5.

Short answer and multiple choice. You do not need to show your work on this problem, and there is no partial credit.

a) (2 points) Let A be a $n \times n$ matrix, and let u, v, w be nonzero vectors in $\mathbf{R}^n$ which are distinct (so $u \neq v$, $u \neq w$, and $v \neq w$). Suppose

$$Au = 2u, \quad Av = 2v, \quad Aw = -w.$$

Which one of the following vectors must be an eigenvector of A?

(i) $u - v$

(ii) $v - w$

(iii) $u - w$

(iv) none of the above

b) (2 points) Circle the matrix A below whose characteristic polynomial is

$$\det(A - \lambda I) = -\lambda^3 - \lambda.$$

(i) $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(iii) $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(iv) $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

c) (3 points) Suppose A is a 3×3 matrix with $\det(A) = -3$, and let T be the matrix transformation $T(x) = Ax$. which of the following must be true? Circle all that apply.

(i) $\det(2A) = -24$.

(ii) If S is a solid with volume 10, then the volume of $T(S)$ is 30.

(iii) $\lambda = -3$ is an eigenvalue of A.

d) (3 points) Suppose W is a subspace of $\mathbf{R}^n$ and P is the matrix for orthogonal projection onto W. Which of the following statements must be true? Circle *all* that apply.

(i) $P^4 = P$.

(ii) $I + P$ is invertible.

(iii) $I - P$ is invertible.

Solution.

a) The correct answer is (i), since $A(u-v) = Au - Av = 2u - 2v = 2(u-v)$ and $u - v \neq 0$. Here (ii) and (iii) are false because they concern eigenvectors in different eigenspaces.

b) The answer is (iv). The c.p. is $-\lambda^3 - \lambda = -\lambda(\lambda^2+1)$. We see (i) and (iii) are false immediately because neither has 0 as an eigenvalue. Also, (ii) is false because its eigenvalues are 0 and 1. This means (iv) must be correct, and it is indeed correct because its c.p. is

$$(0-\lambda)(\lambda^2-(-1)) = -\lambda(\lambda^2+1) = -\lambda^3 - \lambda.$$

c) The correct answers are (i) and (ii). For part (i), $\det(2A) = 2^3 \det(A) = 2^3(-3) = -24$. Part (ii) is true from the standard formula relating volumes and determinants. Part (iii) is false, for example $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has determinant -3 but does not have -3 as an eigenvalue.

d) Parts (i) and (ii) must be true. Since $PP = P^2 = P$, we know $P^4 = P^2P^2 = PP = P$ so (i) is true. Part (ii) is true because -1 is never an eigenvalue of a projection P, so $P - (-I)$ must be invertible. Part (iii) might not be true. For example, take $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ the projection onto the x-axis in $\mathbf{R}^2$. Then $I - P = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ which is not invertible.

Problem 6.

Parts (a), (b), and (c) are unrelated and are 5 points, 2 points, and 3 points, respectively. You do not need to show your work on this page, and there is no partial credit.

a) Suppose W is a subspace of $\mathbf{R}^3$ and x is a vector in $\mathbf{R}^3$ whose orthogonal decomposition with respect to W is $x = x_W + x_{W^\perp}$, where

$$x_W = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad x_{W^\perp} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

(i) What is the closest vector to x in W?

(I) $\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$ (II) $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ (III) $\begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix}$ (IV) $\begin{pmatrix} 4 \\ -5 \\ 0 \end{pmatrix}$

(V) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (VI) $\sqrt{30}\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$ (VII) $\sqrt{11}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

(ii) Find the distance from x to W.

(I) $\sqrt{30}$ (II) $\sqrt{11}$ (III) $\sqrt{41}$ (IV) $\sqrt{51}$

(V) 0 (VI) 30 (VII) 11 (VIII) 5

(iii) Is $\begin{pmatrix} 10 \\ -4 \\ 2 \end{pmatrix}$ in W? YES NO NOT ENOUGH INFORMATION

b) Let $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Let S be the set of all vectors $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $\mathbf{R}^2$ that satisfy $x \cdot u = 3$. Which one of the following describes S?

(i) S consists only of the vector $\begin{pmatrix} 3/5 \\ 6/5 \end{pmatrix}$

(ii) S is the line $x_1 + 2x_2 = 3$ in $\mathbf{R}^2$.

(iii) $S = \text{Span}\left\{\begin{pmatrix} 3/5 \\ 6/5 \end{pmatrix}\right\}$

(iv) $S = \text{Span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$

c) Let $W = \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 - 2x_3 = 0\right\}$, and let B be the matrix for orthogonal projection onto W. Which of the following statements are true? Select *all* that apply.

(i) The eigenvalues of B are $\lambda = 0$ and $\lambda = 1$.

(ii) $B\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(iii) The null space of B is 1-dimensional.

Solution.

a) For part (i), the closest vector is $x_W = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$, which is choice (I).

For part (ii), the answer is

$$||x_{W^\perp}|| = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}.$$

For part (iii), the answer is YES since $\begin{pmatrix} 10 \\ -4 \\ 2 \end{pmatrix} = 2x_W$, which is in W.

b) The answer is (ii): Solving $x \cdot u = 3$ gives $x_1 + 2x_2 = 3$, which is a line in $\mathbf{R}^2$.

c) Parts (i), (ii), and (iii) are all true.

Part (i) uses a standard fact about projections: B is the orthogonal projection onto a 2-dimensional subspace of $\mathbf{R}^3$, so its eigenvalues are 0 and 1 ($Bx = x$ if x is in W and $Bx = 0$ if x is in $W^\perp$).

Part (ii) is true: note $x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is in W since $1 - 1 - 2(0) = 0$, so $Bx = x$.

For part (iii), the null space of B is $W^\perp$. Here, W is a plane ($W = \mathrm{Nul}\begin{pmatrix} 1 & -1 & -2 \end{pmatrix}$) so $W^\perp$ is a line, thus $\dim(\mathrm{Nul}\ B) = \dim(W^\perp) = 1$.

Problem 7.

Free response. Show your work in parts (a) and (e).

Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the transformation such that $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ z \end{pmatrix}$ and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation of rotation counterclockwise by 90 degrees.

a) (3 points) Write the standard matrix A for T. Enter your answer in the space below.

$$A = \begin{pmatrix} & & \\ & & \end{pmatrix}$$

b) (2 points) Write the standard matrix B for U. Enter your answer in the space below.

$$B = \begin{pmatrix} & \\ & \end{pmatrix}$$

c) (1 point) Is T onto? YES NO

d) (1 point) Circle the composition that makes sense: $T \circ U$ $U \circ T$.

e) (3 points) Using matrix multiplication, find the standard matrix C for the composition you chose above. Enter your answer in the space provided below.

$$C = \begin{pmatrix} & \\ & \end{pmatrix}$$

Solution.

a) $A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$

b) $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$

c) Yes, T is onto because A has a pivot in each row.

d) $U \circ T$ makes sense.

e) $C = BA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}.$

Problem 8.

Free response. Show your work! Parts (a) and (b) are unrelated.

a) (6 points) Let $A = \begin{pmatrix} 4 & 5 \\ -2 & -2 \end{pmatrix}$.

Find the eigenvalues of A. For the eigenvalue with *positive* imaginary part, find one corresponding eigenvector v. Enter your answers in the space provided below.

The eigenvalues are ____________________. [simplify the eigenvalues completely]

For the eigenvalue with *positive* imaginary part, an eigenvector is $v = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$.

b) (4 pts) Let $W = \text{Span}\left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$. Find the matrix B for orthogonal projection onto W.

Enter your answer in the space below.

$$B = \begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}$$

Solution.

a) The c.p. of A is

$$\det(A - \lambda I) = \lambda^2 - \text{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 2.$$

The eigenvalues are

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

Now for an eigenvector v for $\lambda = 1 + i$

$$\left(A - (1+i)I \mid 0\right) = \left(\begin{array}{cc|c} 3-i & 5 & 0 \\ -2 & -3-i & 0 \end{array}\right) = \left(\begin{array}{cc|c} a & b & 0 \\ (*) & (*) & 0 \end{array}\right).$$

so one is $v = \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} -5 \\ 3-i \end{pmatrix}$. Or alternatively $v = \begin{pmatrix} b \\ -a \end{pmatrix} = \begin{pmatrix} 5 \\ -3+i \end{pmatrix}$.

There are also other possibilities. For example, had we used the 2×2 eigenvector trick in the second row, we would have gotten $v = \begin{pmatrix} 3+i \\ -2 \end{pmatrix}$. Another possibility is $v = \begin{pmatrix} -3-i \\ 2 \end{pmatrix}$ (which is just -1 times the vector in the previous sentence).

b) This problem was nearly copied from our Chapter 6 worksheet. With $u = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, the projection is

$$W = \frac{1}{u \cdot u} u u^T = \frac{1}{1^2 + 2^2} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Many students attempted to solve an entirely different problem by finding a basis for $W^\perp$. Unfortunately, that has nothing to do with finding B unless you are using the bases of W and $W^\perp$ to find B using diagonalization (and I don't think a single student found B by diagonalization). Many students wrote 3×2 matrices or vectors in $\mathbf{R}^3$ as their answer, but such answers do not make sense at all.

Problem 9.

Free response. Show your work! For this problem, let $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

a) (3 points) Find all eigenvalues of A and write them in the box below.

b) (5 points) For each of the eigenvalues, find a basis of the corresponding eigenspace.

c) (2 points) A is diagonalizable. Write an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

Solution.

a) Using the cofactor expansion along the first row:

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = (2-\lambda)(\lambda^2 - 1) = (2-\lambda)(\lambda - 1)(\lambda + 1).$$

Thus the eigenvalues are $-1,\ 1,\ 2$.

b) For $\underline{\lambda = -1}$:

$$\left(A + I \mid 0\right) = \left(\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives $x_1 = 0$ and $x_2 = -x_3$, so the (-1)-eigenspace is $\operatorname{Span}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$.

For $\underline{\lambda = 1}$:

$$\left(A - I \mid 0\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives $x_1 = 0$ and $x_2 = x_3$, so the 1-eigenspace is $\operatorname{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$.

For $\underline{\lambda = 2}$:

$$\left(A - 2I \mid 0\right) = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives $x_1 = x_3$ and $x_2 = x_3$, so the 2-eigenspace is $\operatorname{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$.

c) Many answers possible, for example $C = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Problem 10.

Free response. Show your work.

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0,10), \qquad (3,7), \qquad (6,-8).$$

Enter your answer below:

$$y = \_\_\_\_\_\_x \; + \; \_\_\_\_\_\_.$$

You must show appropriate work and *simplify your answer completely* (if your answer has fractions, simplify them completely). If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

Solution.

We imagine there was a line through three points.

$x = 0$, $y = 10$: $\quad 10 = M(0) + B \quad 0M + B = 10$
$x = 3$, $y = 7$: $\quad 7 = M(3) + B \quad 3M + B = 7$
$x = 6$, $y = -8$: $\quad -8 = M(6) + B \quad 6M + B = -8$

This is $Ax = b$ where $A = \begin{pmatrix} 0 & 1 \\ 3 & 1 \\ 6 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 10 \\ 7 \\ -8 \end{pmatrix}$.

We will need solve $A^T A\hat{x} = A^T b$.

$$A^T A = \begin{pmatrix} 0 & 3 & 6 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 3 & 1 \\ 6 & 1 \end{pmatrix} = \begin{pmatrix} 45 & 9 \\ 9 & 3 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} 0 & 3 & 6 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 10 \\ 7 \\ -8 \end{pmatrix} = \begin{pmatrix} -27 \\ 9 \end{pmatrix}.$$

Now we solve for $\hat{x}$:

$$\left(A^T A \mid A^T b\right) = \left(\begin{array}{cc|c} 45 & 9 & -27 \\ 9 & 3 & 9 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1/3]{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 3 & 1 & 3 \\ 45 & 9 & -27 \end{array}\right) \xrightarrow{R_2 = R_2 - 15R_1} \left(\begin{array}{cc|c} 3 & 1 & 3 \\ 0 & -6 & -72 \end{array}\right)$$

$$\xrightarrow[R_2 = -(1/6)R_2]{R_1 = R_1/3} \left(\begin{array}{cc|c} 1 & 1/3 & 1 \\ 0 & 1 & 12 \end{array}\right) \xrightarrow{R_1 = R_1 - (1/3)R_2} \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 12 \end{array}\right).$$

This gives $\hat{x} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}$, so

$$y = -3x + 12.$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.