MATH 1553, FALL 2023
SAMPLE MIDTERM 1B: COVERS THROUGH SECTION 2.4

Name	

Please **read all instructions** carefully before beginning.

- Write your name at the top of each page (not just the cover page!).
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. We recommend completing the practice exam in 75 minutes, without notes or distractions.

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Problem 1.

These problems are true or false. Circle **T** if the statement is *always* true.
Otherwise, circle **F**. You do not need to justify your answer, and there is no partial credit.

a) **T** **F** The matrix $\left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ is in reduced row echelon form.

b) **T** **F** A system of 3 linear equations in 4 variables can have exactly one solution.

c) **T** **F** The vector equation $x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is consistent.

d) **T** **F** Suppose A is an 4×3 matrix whose first column is the sum of its second and third columns. Then the equation $Ax = 0$ has infinitely many solutions.

e) **T** **F** If A is an $m \times n$ matrix and $m > n$, then then there is at least one vector b in $\mathbf{R}^m$ which is not in the span of the columns of A.

Solution.

a) True.

b) False. The augmented matrix can have at most 3 pivots, so there will be at least one free variable, thus any associated system will either be inconsistent or have infinitely many solutions.

c) False. The associated augmented matrix has a pivot in the rightmost column.

d) True. From the given conditions, A has 3 columns but its column span cannot be any more than a plane since its first column is in the span of its second and third columns. Therefore, A has fewer than 3 pivots, so at least one column doesn't have a pivot and therefore $Ax = 0$ will have at least one free variable and thus infinitely many solutions.

e) True. The matrix A has at most n pivots, but it has m rows and $m > n$ so it cannot have a pivot in every row.

Problem 2.

Short answer. You do not need to show your work or justify your answer.

a) Complete the following mathematical definition of linear combination (be mathematically precise!): Let $v_1, v_2 \dots, v_p$, and w be vectors in $\mathbf{R}^n$.
We say w is a *linear combination* of $v_1, \dots, v_p$ if...

b) Are there three nonzero vectors $v_1,\ v_2,\ v_3$ in $\mathbf{R}^3$ so that Span$\{v_1,\ v_2,\ v_3\}$ is a plane but v_3 is not in Span$\{v_1, v_2\}$? If your answer is yes, write such vectors $v_1,\ v_2,\ v_3$ and label each vector clearly.

c) Write a matrix A with the property that the equation $Ax = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is consistent.

d) Suppose A is a 2×3 matrix and v is some vector so that the set of solutions to $Ax = v$ has parametric form

$$x_1 = 1 + x_3 \qquad x_2 = 2 - x_3 \qquad x_3 = x_3 \qquad (x_3 \text{ free}).$$

Which of the following must be true? Circle all that apply.

(i) The solution set for $Ax = 0$ is Span $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

(ii) For each b in $\mathbf{R}^2$, the equation $Ax = b$ is consistent.

(iii) v is not the zero vector.

Solution.

a) $w = x_1 v_1 + x_2 v_2 + \cdots + x_p v_p$ for some scalars $x_1, \dots, x_p$.

b) For example, $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

c) All we need is a matrix with 3 rows, so that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is in the span of its columns. For example,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or even} \quad A = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ or } \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

d) We see (i) is true because the homogeneous solution set is the translation of a consistent set by any particular solution (here $(1, 2, 0)$ is a particular solution)

We see (ii) is true because A must have two pivots for the equation $Ax = v$ to have exactly one free variable, which means A has a pivot in every row.

We see (iii) is true because if $v = 0$ then $x = 0$ would be a solution to $Ax = v$, but the solution set to $Ax = v$ doesn't include the origin here (if it did then $x_3 = 0$ but then $x_1 = 1$ and $x_2 = 2$).

Problem 3.

a) Suppose A is a 3×3 matrix and b_1 and b_2 are vectors in $\mathbf{R}^3$. Answer each of the following questions.
(i) Is it possible for $\left(A \mid b_1\right)$ to have a unique solution and $\left(A \mid b_2\right)$ to have infinitely many solutions? YES NO

(ii) Is it possible for $\left(A \mid b_1\right)$ to have a unique solution and $\left(A \mid b_2\right)$ to be inconsisent? YES NO

b) Suppose we are given a consistent linear system of 4 equations in 5 variables, and suppose that the augmented matrix corresponding to the system has 3 pivots. Then the solutions to the system is a:

(circle one answer) point line plane 3-space

in:

(circle one answer) $\mathbf{R}^2$ $\mathbf{R}^3$ $\mathbf{R}^4$ $\mathbf{R}^5$.

c) Suppose that the plane $x_1 - 4x_2 + x_3 = 0$ is the set of solutions to the matrix equation $Ax = 0$, and suppose that $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ is a solution to $Ax = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

(i) Is it true that $A\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$? YES NO

(ii) Is it true that $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ is a solution to $Ax = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$? YES NO

Solution: Part (a): (i) No. A is a 3×3 matrix, and if $\left(A \mid b_1\right)$ has a unique solution, that means there is no free variable, and there are 3 pivots in the coefficient matrix A. So every row and every column of A has exactly one pivot. Then for any b_2, the $\left(A \mid b_2\right)$ will exactly have only one solution.

(ii) No. Same as last part: A is 3×3, and if $\left(A \mid b_1\right)$ has a unique solution, that means there are 3 pivots in the coefficient matrix A. So every row and every column of A has exactly one pivot. for any b_2, the $\left(A \mid b_2\right)$ will exactly have only one solution.

Part (b): Plane in $\mathbf{R}^5$.

Part (c): For (i), We could just observe that $\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$ satisfies

$$x_1 - 4x_2 + x_3 = 4 - 4(1) + 0 = 0, \text{ so } \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} \text{ is a solution to } Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

However, this shortcut is not necessary: The plane $x_1 - 4x_2 + x_3 = 0$ corresponds to $\left(\begin{array}{ccc|c} 1 & -4 & 1 & 0 \end{array}\right)$. This has solution set $x_1 = 4x_2 - x_3$, $x_2 = x_2$, $x_3 = x_3$ (where x_2 and x_3 are free) and in parametric vector form,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, everything in the $\text{Span}\left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a solution to $Ax = 0$. In particular $A\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

(ii) Yes. One way to see this is that since $\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$ is a solution to $Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and we were told $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ is a solution to $Ax = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, their sum $\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ is a solution to $Ax = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ by the theory of solution sets. We could also compute this explicitly if we wanted:

$$A\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = A\left(\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \right) = A\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + A\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

The matrix A that was actually used for this problem is

$$A = \begin{pmatrix} -2/5 & 8/5 & -2/5 \\ -1/5 & 4/5 & -1/5 \\ 0 & 0 & 0 \end{pmatrix}.$$

However, it would be incredibly hard to go backwards to reconstruct A from the problem, so we needed to the information we were given.

d) Find all values of a (if there are any) so that the following matrix is in reduced row echelon form: $\begin{pmatrix} 1 & -2 & -1 \\ 0 & a & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Briefly justify your answer.

Solution: If the matrix has any hope of being in RREF, then a must be 0 (otherwise, a would correspond to a pivot with a nonzero entry above it).

However, if $a = 0$ then the matrix is $\begin{pmatrix} 1 & -2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, which has a nonzero entry above its rightmost pivot and is thus not in RREF.

Therefore, there is no value of a for which the matrix is in RREF.

Problem 4.

Free response. Show all work and justify answers as appropriate.

a) Consider the following system of linear equations.

$$\begin{aligned} x + 2y + \ \ z &= d \\ 3y - 3z &= 6 \\ -2x - 4y + cz &= 7 \end{aligned}$$

(i) For which values of c and d will the linear system be inconsistent?

(ii) For which values of c and d will the linear system have infinitely many solutions? Write the solutions in parametric form with these values of c and d.

b) Find all values of h so that $\begin{pmatrix} 3 \\ -1 \\ h \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

Solution.

a) (i) One step of row-reduction gives the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & d \\ 0 & 3 & -3 & 6 \\ 0 & 0 & c+2 & 7+2d \end{array}\right).$$

In order for the system to be inconsistent, we must have a pivot in the right column of the augmented matrix, so $c + 2 = 0$ and $7 + 2d \neq 0$. Therefore, $c = -2$ and $d \neq -7/2$.

(ii) To have infinitely many solutions, we must have a free variable and the rightmost column cannot have a pivot, so $c = -2$ and $d = -7/2$. The RREF of the augmented matrix in this case is

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & -\frac{15}{2} \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

so x_3 is free and

$$x_1 = -\frac{15}{2} - 3x_3, \qquad x_2 = 2 + x_3, \qquad x_3 = x_3 \quad \text{(any real number)}.$$

b) We row-reduce:

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 1 & 3 & h \end{array}\right) \xrightarrow{R_3 = R_3 - R_1} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 2 & h-3 \end{array}\right) \xrightarrow{R_3 = R_3 - 2R_2} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & h-1 \end{array}\right).$$

This is consistent if and only if there is not a pivot in the rightmost column, so $h - 1 = 0$, thus $h = 1$.

Problem 5.

Free response. Show all work and justify answers as appropriate.

Consider the following linear system of equations in the variables $x_1,\ x_2,\ x_3$:

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 1 \\ 5x_1 - 10x_2 + 12x_3 &= -3 \\ -3x_1 + 6x_2 - 6x_3 &= -3. \\ 2x_1 - 4x_2 + 5x_3 &= -2. \end{aligned}$$

a) Write the augmented matrix corresponding to this system, and put the augmented matrix into RREF.

b) The system is consistent. Write the set of solutions to the system of equations in parametric vector form.

c) Write *one* specific vector that solves the system of equations.

Solution.

a)

$$\left(\begin{array}{rrr|r} 1 & -2 & 2 & 1 \\ 5 & -10 & 12 & -3 \\ -3 & 6 & -6 & -3 \\ 2 & -4 & 5 & -2 \end{array}\right) \xrightarrow[R_3=R_3+3R_1,\, R_4=R_4-2R_1]{R_2=R_2-5R_1} \left(\begin{array}{rrr|r} 1 & -2 & 2 & 1 \\ 0 & 0 & 2 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{array}\right) \xrightarrow[\text{then destroy } R_4]{R_2=R_2/2} \left(\begin{array}{rrr|r} 1 & -2 & 2 & 1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1=R_1-2R_2} \left(\begin{array}{rrr|r} 1 & -2 & 0 & 9 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

b) From (a) we see x_2 is free, and

$$x_1 = 9 + 2x_2, \qquad x_2 = x_2, \qquad x_3 = -4.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 + 2x_2 \\ x_2 \\ -4 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ -4 \end{pmatrix} + \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ -4 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

c) Many examples possible. For example, $\begin{pmatrix} 9 \\ 0 \\ -4 \end{pmatrix}$ or $\begin{pmatrix} 11 \\ 1 \\ -4 \end{pmatrix}$.

Problem 6.

Parts (a) and (b) are unrelated. Show your work and justify your answers.

a) Write an augmented matrix in RREF representing a system of three equations in two unknowns, whose solution set is the line $x_2 = 2x_1$ in $\mathbf{R}^2$.

b) Let $A = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$. Draw the span of the columns of A below.

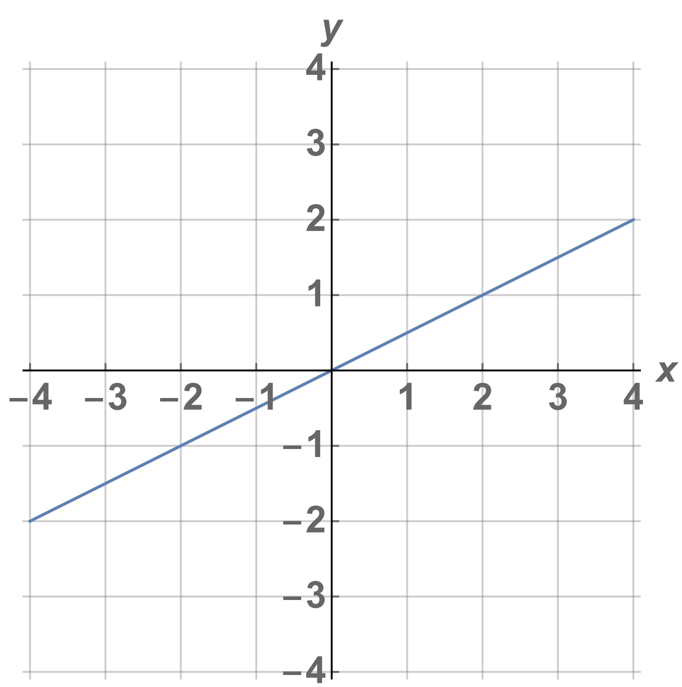
Solution.

a) We need the left side of the augment to be 3×2. Since the solution set includes the origin, the right side of the augment must be the zero vector.

Note $x_2 = 2x_1$ means that $2x_1 - x_2 = 0$, and this corresponds to $\left(\begin{array}{cc|c} 2 & -1 & 0 \end{array}\right)$. However, we need our matrix to be in RREF, so we need to divide by 2 to get the first row of our augmented matrix: $\left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \end{array}\right)$.

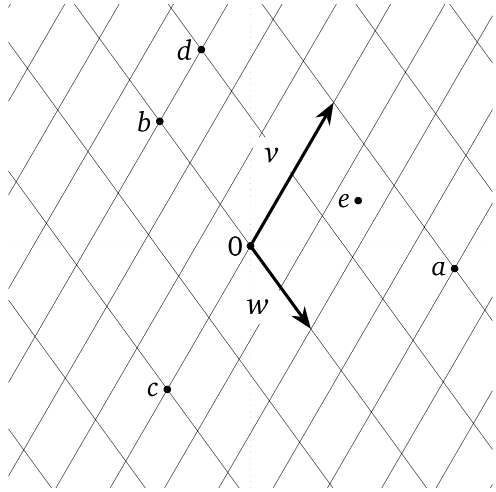
$$\left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

b) The first and second columns are both scalar multiples of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ so the column span is just the span of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, which is the line through the origin and $(2, 1)$.



Problem 7.

Consider the following picture involving the two vectors v and w. The gridlines are set up so that we can easily see how to take precise linear combinations of v and w.



a) For each of the labeled points, estimate the coefficients x, y such that the linear combination $xv + yw$ is the vector ending at that point.

$$\_\_\_\_\_\_ \, v + \_\_\_\_\_\_ \, w = a$$

$$\_\_\_\_\_\_ \, v + \_\_\_\_\_\_ \, w = b$$

$$\_\_\_\_\_\_ \, v + \_\_\_\_\_\_ \, w = c$$

$$\_\_\_\_\_\_ \, v + \_\_\_\_\_\_ \, w = d$$

$$\_\_\_\_\_\_ \, v + \_\_\_\_\_\_ \, w = e$$

b) Find two vectors p, q in $\mathbf{R}^2$ such that *none* of the points a, b, c, d, e is in $\mathrm{Span}\{p, q\}$.

You don't need to show your work in this problem.

Solution.

a) As you can tell from the grid, you reach a by following v once then w twice. Hence $a = v + 2w$. Similarly, $b = 0v - \frac{3}{2}w$, $c = -v + 0w$, $d = \frac{1}{2}v - \frac{3}{2}w$, and $e = \frac{3}{4}v + \frac{3}{4}w$.

b) None of the vectors a, b, c, d, e is contained in the x-axis. Therefore they are not contained in

$$\mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}.$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.