MATH 1553, SPRING 2024

SAMPLE MIDTERM 3B: COVERS 4.1 THROUGH 5.6

Name		**GT ID**	

Circle your lecture below.

Jankowski (A and HP, 8:25-9:15 AM) Jankowski (G, 12:30-1:20 PM)

Hausmann (I, 2:00-2:50 PM) Sanchez-Vargas (M, 3:30-4:20 PM)

Athanasouli (N and PNA, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. We recommend completing the practice exam in 75 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §4.1 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§4.1 through 5.6.

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Problem 1.

True or false. Circle **T** if the statement is *always* true.
Otherwise, circle **F**. You do not need to show work or justify your answer.

a) **T** **F** If A is an $n \times n$ matrix, then the determinant of A is the same as the determinant of the RREF of A.

b) **T** **F** If A is a 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = (1 - \lambda)(-1 - \lambda)^2,$$

then A must be invertible.

c) **T** **F** Suppose A is an $n \times n$ matrix and λ is an eigenvalue of A. If v and w are two different eigenvectors of A corresponding to the eigenvalue λ, then $v - w$ is an eigenvector of A.

d) **T** **F** If A and B are 3×3 matrices that have the same eigenvalues and the same algebraic multiplicity for each eigenvalue, then $A = B$.

e) **T** **F** Let $A = \begin{pmatrix} 0.3 & 0.5 & 0.4 \\ 0.3 & 0.1 & 0.4 \\ 0.4 & 0.4 & 0.2 \end{pmatrix}$. Then the 1-eigenspace of A is a line.

Solution.

a) False. If this were true then every invertible matrix would have determinant 1 since the RREF of any invertible matrix is the identity matrix.

b) True. From the characteristic polynomial we see that 0 is not an eigenvalue of A.

c) True: $A(v - w) = Av - Aw = \lambda v - \lambda w = \lambda(v - w)$, and $v - w \neq 0$ since the statement says $v \neq w$.

d) False. For example, $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and I.

e) True: It is quick to check that A is positive stochastic, so its 1-eigenspace is a line by the Perron-Frobenius Theorem, namely the line spanned by its steady-state vector. It is not necessary to do any row-reduction in this problem.

Problem 2.

You do not need to show your work or justify your answers.

a) Complete the following definition (be mathematically precise!):
Suppose A is an $n \times n$ matrix and λ is a real number. We say λ is an *eigenvalue* of A if...

b) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 3$. Find $\det\begin{pmatrix} -4a+d & -4b+e & -4c+f \\ a & b & c \\ g & h & i \end{pmatrix}$.

c) Write a 2×2 matrix which is neither diagonalizable nor invertible.

d) Suppose A is an $n \times n$ matrix and $\det(A) = 0$.
Which of the following statements must be true? Circle all that apply.

(i) $\dim(\mathrm{Nul}(A)) \geq 1$.

(ii) The equation $Ax = 0$ has only the trivial solution $x = 0$.

(iii) $\lambda = 0$ is an eigenvalue of A.

(iv) The equation $Ax = b$ must be inconsistent for at least one b in $\mathbf{R}^n$.

Solution.

a) ... the equation $Ax = \lambda x$ has a non-trivial solution.

b) The matrix is obtained from the original by swapping the first two rows and then doing a row replacement, so the determinant is $(-1)(3) = -3$.

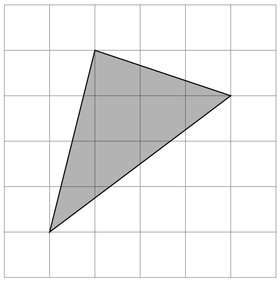
c) Many examples possible. For example, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

d) (i), (iii), and (iv) are true, but (ii) is not. Since $\det(A) = 0$ we know A is not invertible, which means it has more than just the zero vector in its nullspace, and $\lambda = 0$ is an eigenvalue, and the transformation $T(x) = Ax$ is not onto.

Problem 3.

Short answer. You do not need to show your work unless you are told to do so.

a) Find the area of the triangle given below (the grid marks are spaced one unit apart). Briefly show your work.



b) Suppose A is a 3×3 matrix and its characteristic polynomial is

$$\det(A - \lambda I) = -(\lambda - 5)(\lambda - 3)^2.$$

Which of the following *must* be true? Circle all that apply.

(i) The 5-eigenspace of A has dimension 1.

(ii) If the 3-eigenspace of A is the xy-plane, then A is diagonalizable.

(iii) $\det(A) = 45$.

(iv) The homogeneous system given by the equation $(A - 3I)x = 0$ has two free variables.

c) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects across the line $y = -2x$, and let A be the standard matrix for T. Which of the following are true? Circle all that apply.

(i) The 1-eigenspace of A is Span $\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$.

(ii) A is diagonalizable.

(iii) $\det(A + I) = 0$.

Solution.

a) The vectors starting at the bottom left and going to the other vertices are $(4,3)$ and $(1,4)$, so the area is

$$\frac{1}{2}\left|\det\begin{pmatrix} 4 & 1 \\ 3 & 4 \end{pmatrix}\right| = \frac{1}{2}(16-3) = \frac{13}{2}.$$

b) We see (i) is true because 5 has algebraic multiplicity 1 thus has geometric multiplicity 1.

For (ii), if the 3-eigenspace of A is the xy-plane then A is diagonalizable because then A is a 3×3 matrix with sum of geometric multiplicities equal to $2+1=3$.

For (iii), $\det(A) = \det(A - 0I) = -(-5)(-3)^2 = 45$. However (iv) is not necessarily true, for example

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

c) All of them are true! We see (i) is true since T fixes all vectors along the line $y = -2x$, and (ii) is true because A is a 2×2 matrix with two distinct real eigenvalues 1 and -1. Also, (iii) is true because -1 is an eigenvalue of A (recall that A flips all vectors perpendicular to the line $y = -2x$).

Problem 4.

Short answer. Show your work on part (c).

a) Suppose A is a positive stochastic matrix and $A^n\begin{pmatrix}390\\110\end{pmatrix}$ approaches $\begin{pmatrix}100\\400\end{pmatrix}$ as $n\to\infty$. What is the steady-state vector for A?

b) Suppose $A=\begin{pmatrix}1&2\\-1&3\end{pmatrix}\begin{pmatrix}-1/3&0\\0&1\end{pmatrix}\begin{pmatrix}1&2\\-1&3\end{pmatrix}^{-1}$. Which of the following are true? Circle all that apply.

(i) Every nonzero vector in $\mathbf{R}^2$ is an eigenvector of A.

(ii) Repeated multiplication by A pushes vectors toward Span$\left\{\begin{pmatrix}2\\3\end{pmatrix}\right\}$.

(iii) If $x=\begin{pmatrix}-1\\1\end{pmatrix}$, then $A^n x$ approaches the zero vector as n becomes very large.

(iv) The eigenvalues of A are $-\dfrac{1}{3}$ and 1.

c) Let $A=\begin{pmatrix}0.5&0.3\\0.5&0.7\end{pmatrix}$. As k goes to infinity, what vector does $A^k\begin{pmatrix}8\\0\end{pmatrix}$ approach?

Solution.

a) $A^n\begin{pmatrix}390\\110\end{pmatrix}$ approaches 500 times the steady-state vector as $n\to\infty$, which is the vector $\begin{pmatrix}100\\400\end{pmatrix}$, so the steady state vector is

$$\frac{1}{500}\begin{pmatrix}100\\400\end{pmatrix}=\begin{pmatrix}1/5\\4/5\end{pmatrix}.$$

b) Statements (ii), (iii), and (iv) are true. Write $v_1=\begin{pmatrix}1\\-1\end{pmatrix}$ and $v_2=\begin{pmatrix}2\\3\end{pmatrix}$. We see A has eigenvalues $\lambda=-\dfrac{1}{3}$ and $\lambda=1$ since $Av_1=-\frac{v_1}{3}$ and $Av_2=v_2$, so (iv) is true.

If v is any vector in $\mathbf{R}^2$, then taking Av shrinks the v_1-component of v by a factor of 1/3 (and flips it) and fixes the v_2-component of v, thereby pushing every vector towards the 1-eigenspace Span$\{v_2\}$ and in fact fixing every vector in Span$\{v_2\}$.

In (iii), $A^n x=A^n(-v)=\left(\frac{-1}{3}\right)^n(-v)$ which approaches the origin as $n\to\infty$.

However (i) is not true: for example,

$$A\begin{pmatrix}3\\2\end{pmatrix}=A\left(\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}2\\3\end{pmatrix}\right)=\begin{pmatrix}-1/3\\1/3\end{pmatrix}+\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}5/3\\10/3\end{pmatrix}$$

which is not a scalar multiple of $\begin{pmatrix}3\\2\end{pmatrix}$.

c) $A^k \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ approaches $\boxed{\begin{pmatrix} 3 \\ 5 \end{pmatrix}}$.

Since A is a positive stochastic matrix, the Perron-Frobenius Theorem says that $A^k \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ approaches $(8+0)w$, where w is the steady-state vector for A. To get w, we find a 1-eigenvector and then scale:

$$\left(A - I \mid 0\right) = \left(\begin{array}{cc|c} -0.5 & 0.3 & 0 \\ 0.5 & -0.3 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -3/5 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore, the 1-eigenspace is spanned by $v = \begin{pmatrix} 3/5 \\ 1 \end{pmatrix}$, so

$$w = \frac{1}{1+\frac{3}{5}} \begin{pmatrix} 3/5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/8 \\ 5/8 \end{pmatrix}, \qquad \text{so } A^k \text{ approaches} \quad 8w = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

Problem 5.

Parts (a) and (b) are unrelated.

a) Let $A = \begin{pmatrix} 5 & 5 \\ -2 & -1 \end{pmatrix}$. Find the complex eigenvalues of A. For the eigenvalue with positive imaginary part, find one corresponding eigenvector.
Calculations show the characteristic equation of A is $\lambda^2 - 4\lambda + 5 = 0$.

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i.$$

For $\lambda = 2 + i$ we have

$$\left(A - (2+i) \mid 0\right) = \left(\begin{array}{cc|c} 5-(2+i) & 5 & 0 \\ & * & 0 \end{array}\right) = \left(\begin{array}{cc|c} 3-i & 5 & 0 \\ & * & 0 \end{array}\right).$$

Using the familiar quick trick from class we get $v = \begin{pmatrix} -5 \\ 3-i \end{pmatrix}$. Alternative methods give equivalent correct eigenvectors like $\begin{pmatrix} -3-i \\ 2 \end{pmatrix}$ and $\begin{pmatrix} \frac{-3-i}{2} \\ 1 \end{pmatrix}$, which are complex scalar multiples of the v we wrote.

b) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & c & c & 1 \\ 3 & 0 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Find all values of c so that $\det(A) = 4$.
Cofactor expansion along the 3rd column gives

$$\det(A) = c(-1)^{2+3} \det \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 4 \\ 1 & 0 & 1 \end{pmatrix} = -c\left(1(0) - 2(-1) + 3(0)\right) = -2c.$$

Thus $4 = -2c$, so $c = -2$.

Problem 6.

Let $A = \begin{pmatrix} 3 & 0 & -2 \\ 2 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

a) Find all eigenvalues of A.

Cofactor expansion along the 3rd row gives us

$$\det(A - \lambda I) = (1-\lambda)[(3-\lambda)(1-\lambda)],$$

so the eigenvalues are $\lambda = 1$ and $\lambda = 3$.

b) Find a basis for each eigenspace of A.

$$\left(A - I \mid 0\right) = \left(\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so $x_1 = x_3$ and x_2 and x_3 are free. The 1-eigenspace has basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

$$\left(A - 3I \mid 0\right) = \left(\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 2 & -2 & -2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

so $x_1 = x_2$ and x_2 is free, and $x_3 = 0$. The 3-eigenspace has basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

c) Is A diagonalizable? If so, write an invertible 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$. If not, justify why A is not diagonalizable.

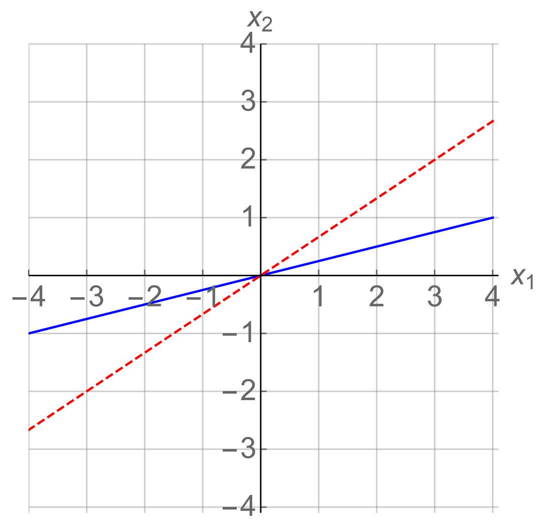
Since A has three linearly independent eigenvectors, it is diagonalizable: $A = CDC^{-1}$ where C consists of eigenvectors and D consists of the corresponding eigenvalues in the appropriate order.

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Problem 7.

Parts (a), (b), and (c) are unrelated.

a) Consider the matrix A whose 1-eigenspace is the solid blue line below and whose 2-eigenspace is the dotted red line below. Find $A\begin{pmatrix}7\\3\end{pmatrix}$.



b) Suppose A and B are 4×4 matrices satisfying

$$\det(A) = 5, \quad \det(AB^{-1}) = 10.$$

Find $\det(-2B)$. Simplify your answer completely.

c) Find all values of c so that the following matrix has exactly one real value of algebraic multiplicity 2.

$$B = \begin{pmatrix} 2 & c \\ -c & 8 \end{pmatrix}.$$

Solution.

a) We write $\begin{pmatrix}7\\3\end{pmatrix}$ as a linear comb. of eigenvectors of A, finding $\begin{pmatrix}7\\3\end{pmatrix} = \begin{pmatrix}4\\1\end{pmatrix} + \begin{pmatrix}3\\2\end{pmatrix}$.

$$A\left(\begin{pmatrix}4\\1\end{pmatrix} + \begin{pmatrix}3\\2\end{pmatrix}\right) = A\begin{pmatrix}4\\1\end{pmatrix} + A\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}4\\1\end{pmatrix} + 2\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}10\\5\end{pmatrix}.$$

b) We compute

$$\det(A)\det(B^{-1}) = 10, \quad \det(B^{-1}) = \frac{10}{\det(A)} = \frac{10}{5} = 2, \qquad \det(B) = \frac{1}{2}.$$

Therefore, $\det(-2B) = (-2)^4 \det(B) = 16 \cdot \frac{1}{2} = 8.$

c) The characteristic polynomial is

$$\lambda^2 - 10\lambda + 16 + c^2.$$

For this to be a perfect square, it must be

$$(\lambda - 5)^2 = \lambda^2 - 10\lambda + 25.$$

Therefore, $16 + c^2 = 25$, so $c = \pm 3$.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.