Math 1553 Worksheet §5.4-5.6

1. True or false. Justify your answer.

A 3×3 matrix A can have a non-real complex eigenvalue with multiplicity 2.

Solution.

No. If c is a (non-real) complex eigenvalue with multiplicity 2, then its conjugate $\bar{c}$ is an eigenvalue with multiplicity 2 since complex eigenvalues always occur in conjugate pairs. This would mean A has a characteristic polynomial of degree 4 or more, which is impossible since A is 3×3.

2. Let $A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}^{-1}$, and let $x = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. What happens to $A^n x$ as n gets very large?

Solution.

We are given diagonalization of A, which gives us the eigenvalues and eigenvectors.

$$\begin{aligned} A^n x &= A^n \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right) = A^n \begin{pmatrix} 2 \\ -1 \end{pmatrix} + A^n \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= 1^n \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \left(\frac{1}{2} \right)^n \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{3}{2^n} \\ \frac{1}{2^n} \end{pmatrix}. \end{aligned}$$

As n gets very large, the entries in the second vector above approach zero, so $A^n x$ approaches $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. For example, for $n = 15$,

$$A^{15} x \approx \begin{pmatrix} 2.00009 \\ -0.999969 \end{pmatrix}.$$

3. Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Find all eigenvalues of A. For each eigenvalue, find an associated eigenvector.

Solution.

The characteristic polynomial is

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 5$$

$$\lambda^2 - 2\lambda + 5 = 0 \iff \lambda = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

For the eigenvalue $\lambda = 1-2i$, we use the shortcut trick you may have seen in class: the first row $\begin{pmatrix} a & b \end{pmatrix}$ of $A-\lambda I$ will lead to an eigenvector $\begin{pmatrix} -b \\ a \end{pmatrix}$ (or equivalently, $\begin{pmatrix} b \\ -a \end{pmatrix}$ if you prefer).

$$\left(A-(1-2i)I \mid 0\right) = \left(\begin{array}{cc|c} 2i & 2 & 0 \\ (*) & (*) & 0 \end{array}\right) \implies v = \begin{pmatrix} -2 \\ 2i \end{pmatrix}.$$

From the correspondence between conjugate eigenvalues and their eigenvectors, we know (without doing any additional work!) that for the eigenvalue $\lambda = 1+2i$, a corresponding eigenvector is $w = \overline{v} = \begin{pmatrix} -2 \\ -2i \end{pmatrix}$.

If you used row-reduction for finding eigenvectors, you would find $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$ as an eigenvector for eigenvalue $1-2i$, and $w = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ as an eigenvector for eigenvalue $1+2i$.

4. A video game offers participants the chance to play as one of two characters: Archer or Barbarian. The game has 100 million players.

In 2023:
Archer is played by 60 million players.
Barbarian is played by 40 million players.

One year later, in 2024:

- 60% of the people who started with the Archer still play with the Archer, while 40% have switched to Barbarian.
- 70% of the customers who stared with the Barbarian still play with the Barbarian, while 30% have switched to Archer.

a) Write down the stochastic matrix A which represents the change in each character's popularity from 2023 to 2024, and use it to find the number of people who played with each character in 2024.

b) Suppose the trend continues each year. In the distant future, who will be the most popular character? What will be the player distribution?

Solution.

a)

$$A = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}, \qquad A\begin{pmatrix} 60 \\ 40 \end{pmatrix} = \begin{pmatrix} 48 \\ 52 \end{pmatrix}.$$

Reminder that the entries in each column of a stochastic matrix sum up to 1.

This means that, in 2024: the archer and barbarian will have 52 million and 48 million players (respectively).

b) To figure out the most popular character over a long period of time, we need to calculate the steady-state for this stochastic matrix, which spans the 1-eigenspace. Solving for $(A-I)v=0$ yields

$$\left(\begin{array}{cc|c} -0.4 & 0.3 & 0 \\ 0.4 & -0.3 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

so the 1-eigenspace is the span of $\begin{pmatrix} 3/4 \\ 1 \end{pmatrix}$ and the steady-state vector for A is

$$\begin{pmatrix} \frac{3/4}{1+3/4} \\ \frac{1}{1+3/4} \end{pmatrix} = \begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix}$$

Thus, in the long-term, about 3/7 of the players will use the archer, while 4/7 of the players will use the barbarian. The player base is 100 million, so eventually the distribution of players will approximately be the following:

$$\text{Archer}: \frac{3}{7}(100) \approx 42.86 \text{ million}$$

$$\text{Barbarian}: \frac{4}{7}(100) \approx 57.14 \text{ million}$$

In the long run, the barbarian will be the most popular character.