Best Work #1: Selected Materials from Math 1553, Introduction to Linear Algebra

Math 1553 Worksheet: Fundamentals and §1.1

Solutions

1. For each equation, determine whether the equation is linear or non-linear. Circle your answer. If the equation is non-linear, briefly justify why it is non-linear.

a) $3x_1 + \sqrt{x_2} = 4$ Linear Not linear

b) $x^2 + y^2 = z$ Linear Not linear

c) $e^{\pi}x + \ln(13)y = \sqrt{2} - z$ Linear Not linear

Solution.

a) Not linear. The $\sqrt{x_2}$ term makes it non-linear.

b) Not linear. It has quadratic terms x^2 and y^2.

c) Linear. Don't be fooled: e^{π} and $\ln(13)$ are just the coefficients for x and y, respectively, and $\sqrt{2}$ is a constant term.

If, for example, the second term had been $\ln(13y)$ instead of $\ln(13)y$, then y would have been inside the logarithm and the equation would have been non-linear.

2. Consider the following three planes, where we use (x, y, z) to denote points in $\mathbf{R}^3$:

$$\begin{aligned} 2x + 4y + 4z &= 1 \\ 2x + 5y + 2z &= -1 \\ y + 3z &= 8 \end{aligned}$$

Do all three of the planes intersect? If so, do they intersect at a single point, a line, or a plane?

Solution.

Subtracting the first equation from the second gives us

$$\begin{aligned} 2x + 4y + 4z &= 1 \\ y - 2z &= -2 \\ y + 3z &= 8. \end{aligned}$$

Next, subtracting the second equation from the third gives us

$$\begin{aligned} 2x + 4y + 4z &= 1 \\ y - 2z &= -2 \\ 5z &= 10, \end{aligned}$$

so $z = 2$. We can back-substitute to find y and then x. The second equation is $y - 2z = -2$, so $y - 2(2) = -2$, thus $y = 2$. The first equation is $2x + 4(2) + 4(2) = 1$, so $2x = -15$, thus $x = -15/2$. We have found that the planes intersect at the point

$$\left(-\frac{15}{2},\ 2,\ 2\right).$$

An alternative method would have been to use augmented matrices to isolate z and then back-substitute:

$$\left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 2 & 5 & 2 & -1 \\ 0 & 1 & 3 & 8 \end{array}\right) \xrightarrow{R_2=R_2-R_1} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 3 & 8 \end{array}\right) \xrightarrow{R_3=R_3-R_2} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 5 & 10 \end{array}\right)$$

The last line is $5z = 10$, so $z = 2$. From here, back-substitution would give us $y = 2$ and then $x = -\frac{15}{2}$, just like before.

3. Find all values of h so that the lines $x + hy = -5$ and $2x - 8y = 6$ do *not* intersect. For all such h, draw the lines $x + hy = -5$ and $2x - 8y = 6$ to verify that they do not intersect.

Solution.

We can use basic algebra, row operations, or geometric intuition.

Using basic algebra: Let's see what happens when the lines *do* intersect. In that case, there is a point (x, y) where

$$\begin{aligned} x + hy &= -5 \\ 2x - 8y &= 6. \end{aligned}$$

Subtracting twice the first equation from the second equation gives us

$$\begin{aligned} x + \qquad hy &= -5 \\ (-8-2h)y &= 16. \end{aligned}$$

If $-8-2h = 0$ (so $h = -4$), then the second line is $0 \cdot y = 16$, which is impossible. In other words, if $h = -4$ then we cannot find a solution to the system of two equations, so the two lines *do not* intersect.

On the other hand, if $h \neq -4$, then we can solve for y above:

$$(-8-2h)y = 16 \qquad y = \frac{16}{-8-2h} \qquad y = \frac{8}{-4-h}.$$

We can now substitute this value of y into the first equation to find x at the point of intersection:

$$x + hy = -5 \qquad x + h \cdot \frac{8}{-4-h} = -5 \qquad x = -5 - \frac{8h}{-4-h}.$$

Therefore, the lines fail to intersect if and only if $\boxed{h = -4}$.

Using intuition from geometry in $\mathbf{R}^2$: Two non-identical lines in $\mathbf{R}^2$ will fail to intersect, if and only if they are parallel. The second line is $y = \frac{1}{4}x - \frac{3}{4}$, so its slope is $\frac{1}{4}$. If $h \neq 0$, then the first line is $y = -\frac{1}{h}x - \frac{5}{h}$, so the lines are parallel when $-\frac{1}{h} = \frac{1}{4}$, which means $h = -4$. In this case, the lines are $y = \frac{1}{4}x + \frac{5}{4}$ and $y = \frac{1}{4}x - \frac{3}{4}$, so they are parallel non-intersecting lines.

(If $h = 0$ then the first line is vertical and the two lines intersect when $x = -5$).

Using row operations: The problem could be done using augmented matrices, which will soon become our main method for solving systems of equations.

$$\left(\begin{array}{cc|c} 1 & h & -5 \\ 2 & -8 & 6 \end{array}\right) \xrightarrow{R_2=R_2-2R_1} \left(\begin{array}{cc|c} 1 & h & -5 \\ 0 & -8-2h & 16 \end{array}\right).$$

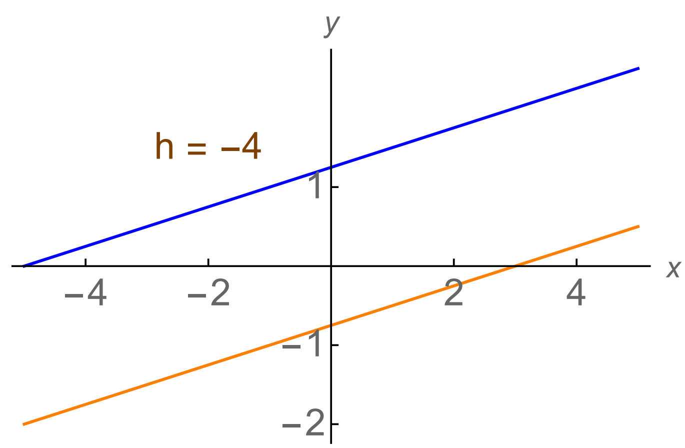
If $-8-2h=0$ (so $h=-4$), then the second equation is $0=16$, so our system has no solutions. In other words, the lines do not intersect.

If $h \neq -4$, then the second equation is $(-8-2h)y=16$, so

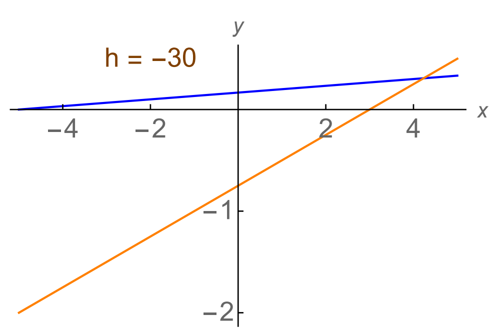
$$y=\frac{16}{-8-2h}=\frac{8}{-4-h} \quad \text{and} \quad x=-5-hy=-5-\frac{8h}{-4-h},$$

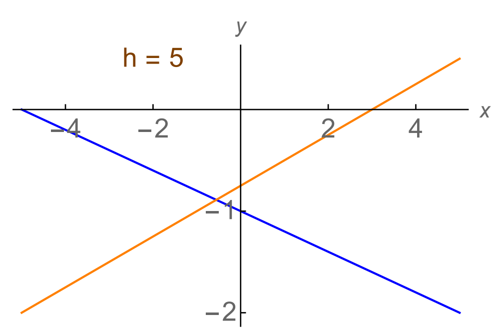
and the lines intersect at (x,y). Therefore, our answer is $h=-4$.

Here are the two lines for $h=-4$, and we can see they are different parallel lines.



If we vary h away from -4, then the blue and orange lines will have different slopes and will inevitably intersect. For example,





Math 1553 Worksheet §1.2, §1.3

Solutions

1. Is it possible for a linear system to have a unique solution if it has more equations than variables? If yes, give an example. If no, justify why it is impossible.

Solution.

It is possible. One example is the system below, which has unique solution $x = 5$, $y = 2$:

$$\begin{aligned} x + y &= 7 \\ x - y &= 3 \\ 2x + 2y &= 14. \end{aligned}$$

2. **a)** Which of the following matrices are in row echelon form? Which are in reduced row echelon form?

b) For the matrices in row echelon form, which entries are the pivots? What are the pivot columns?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

Solution.

The first is in reduced row echelon form; the second is in row echelon form. The pivots are in red; the other entries in the pivot columns are in blue.

3. Find the parametric form of the solutions of following system of equations in x_1, x_2, and x_3 by putting an augmented matrix into reduced row echelon form. State which variables (if any) are free variables. Describe the solution set geometrically.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ - 3x_2 - 6x_3 &= -3. \end{aligned}$$

Solution.

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{array}\right) \xrightarrow{R_2=R_2+4R_1} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{array}\right)$$

$$\xrightarrow{R_3=R_3+R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1=R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2=R_2\div 3} \left(\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The variables x_1 and x_2 correspond to pivot columns, but x_3 is free.

$$x_1 = -2 + 5x_3, \qquad x_2 = 1 - 2x_3, \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

This consistent system in three variables has one free variable, so the solution set is a line in $\mathbf{R}^3$.

Math 1553 Worksheet §2.1, §2.2

Solutions

1. Write a set of three vectors whose span is a plane in $\mathbf{R}^3$.

Solution.

Just choose any two vectors that span your favorite plane, then pick your third vector to be within that plane.

For example, choose $v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, and $v_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}$.

2. Consider the system of linear equations

$$\begin{aligned} x + 2y &= 7 \\ 2x + y &= -2 \\ -x - y &= 4. \end{aligned}$$

Question: Does this system have a solution? If so, what is the solution set?

a) Formulate this question as an augmented matrix.

b) Formulate this question as a vector equation.

c) What does this mean in terms of spans?

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

Solution.

a) As an augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & 1 & -2 \\ -1 & -1 & 4 \end{array}\right)$$

b) What are the solutions to the following vector equation?

$$x\begin{pmatrix}1\\2\\-1\end{pmatrix} + y\begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}7\\-2\\4\end{pmatrix}$$

c) Is $\begin{pmatrix}7\\-2\\4\end{pmatrix}$ in $\text{Span}\left\{\begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\-1\end{pmatrix}\right\}$?

e) Row reducing yields

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right),$$

so there are no solutions. (This should be obvious from the picture in (d)).

3. Paul Drake has challenged you to find his hidden treasure, located at some point (a,b,c). He has honestly guaranteed you that the treasure can be found by starting at the origin and taking steps in directions given by

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} \qquad v_3 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

By decoding Paul's message, you have discovered that the first and second coordinates of the treasure's location are (in order) -4 and 3.

a) What is the treasure's full location?

b) Give instructions for how to find the treasure by only moving in the directions given by v_1, v_2, and v_3.

Solution.

a) We translate this problem into linear algebra. Let c be the final entry of the treasure's location. Since Paul has assured us that we can find the treasure using the three vectors we have been given, our problem is to find c so that $\begin{pmatrix} -4 \\ 3 \\ c \end{pmatrix}$ is a linear combination of v_1, v_2, and v_3 (in other words, find c so that the treasure's location is in in $\text{Span}\{v_1, v_2, v_3\}$). We form an augmented matrix and find when it gives a consistent system.

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & c \end{array}\right) \xrightarrow[R_3=R_3+2R_1]{R_2=R_2+R_1} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & c-8 \end{array}\right) \xrightarrow{R_3=R_3-3R_2} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & c-5 \end{array}\right).$$

This system will be consistent if and only if the right column is not a pivot column, so we need $c-5=0$, or $c=5$.

The location of the treasure is $(-4,3,5)$.

b) Getting to the point $(-4,3,5)$ using the vectors v_1, v_2, and v_3 is equivalent to finding scalars x_1, x_2, and x_3 so that

$$\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

We can rewrite this as

$$\begin{aligned} x_1 + 5x_2 - 3x_3 &= -4 \\ -x_1 - 4x_2 + x_3 &= 3 \\ -2x_1 - 7x_2 &= 5. \end{aligned}$$

We put the matrix from part (a) into RREF.

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1=R_1-5R_2} \left(\begin{array}{ccc|c} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Note x_3 is the only free variable, so:

$$x_1 = 1 - 7x_3, \qquad x_2 = -1 + 2x_3 \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

Since the system has infinitely many solutions, there are infinitely many ways to get to the treasure. If we choose the path corresponding to $x_3 = 0$, then $x_1 = 1$ and $x_2 = -1$, which means that we move 1 unit in the direction of v_1 and -1 unit in the direction of v_2. In equations:

$$\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + 0 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

Math 1553 Worksheet §§2.3-2.5

Solutions

For problems 1 and 2 below: The professor in your widgets and gizmos class is trying to decide between three different grading schemes for computing your final course grade. The schemes are based on homework (HW), quiz grades (Q), midterms (M), and a final exam (F). The three schemes can be described by the following matrix A:

$$\begin{array}{r c} & \begin{array}{cccc} \text{HW} & \text{Q} & \text{M} & \text{F} \end{array} \\ \begin{array}{r} \text{Scheme 1} \\ \text{Scheme 2} \\ \text{Scheme 3} \end{array} & \begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \end{array}$$

Feel free to use a calculator to carry out arithmetic in problems 1 and 2.

1. Suppose that you have a score of x_1 on homework, x_2 on quizzes, x_3 on midterms, and x_4 on the final, with potential final course grades of b_1, b_2, b_3. Write a matrix equation $Ax = b$ to relate your final grades to your scores.

Solution.

In the above grading schemes, you would receive the following final grades:

$$\begin{aligned} &\text{Scheme 1:} \quad 0.1x_1 + 0.1x_2 + 0.5x_3 + 0.3x_4 = b_1 \\ &\text{Scheme 2:} \quad 0.1x_1 + 0.1x_2 + 0.4x_3 + 0.4x_4 = b_2 \\ &\text{Scheme 3:} \quad 0.1x_1 + 0.1x_2 + 0.6x_3 + 0.2x_4 = b_3 \end{aligned}$$

This is the same as the matrix equation

$$(*) \qquad \begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

2. Suppose that you end up with averages of 90% on the homework, 90% on quizzes, 85% on midterms, and a 95% score on the final exam. Use Problem 1 to determine which grading scheme leaves you with the highest overall course grade.

Solution.

According to equation (*) above, your final grades would be

$$\begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \begin{pmatrix} .90 \\ .90 \\ .85 \\ .95 \end{pmatrix} = \begin{pmatrix} .89 \\ .90 \\ .88 \end{pmatrix}.$$

Hence the second grading scheme gives you the best final grade.

3. **a)** True or false. Justify your answer:
If A is a 5×4 matrix, then the equation $Ax = b$ must be inconsistent for some b in $\mathbf{R}^5$.

TRUE **FALSE**

b) Suppose $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ and $A\begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Must it be true that $\{v_1, v_2, v_3\}$ is linearly dependent? If so, write a linear dependence relation for the vectors.

Solution.

a) True. If A is a 5×4 matrix, then A can have at most 4 pivots (since no row or column can have more than 1 pivot). But A has 5 rows, so this means A cannot have a pivot in each row, and therefore $Ax = b$ must be inconsistent for at least one b in $\mathbf{R}^5$.

b) Yes. By definition of matrix multiplication, $-3v_1 + 2v_2 + 7v_3 = 0$, so $\{v_1, v_2, v_3\}$ is linearly dependent and the equation gives a linear dependence relation.

4. Find the solution sets of $x_1 - 3x_2 + 5x_3 = 0$ and $x_1 - 3x_2 + 5x_3 = 3$ and write them in parametric vector form. How do the solution sets compare geometrically?

Solution.

The equation $x_1 - 3x_2 + 5x_3 = 0$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 0 \end{array}\right)$ which has two free variables x_2 and x_3.

$$x_1 = 3x_2 - 5x_3 \qquad x_2 = x_2 \qquad x_3 = x_3.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{x_2\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

The solution set for $x_1 - 3x_2 + 5x_3 = 0$ is the plane spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.

The equation $x_1 - 3x_2 + 5x_3 = 3$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 3 \end{array}\right)$ which has two free variables x_2 and x_3.

$$x_1 = 3 + 3x_2 - 5x_3 \qquad x_2 = x_2 \qquad x_3 = x_3.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 + 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

This solution set is the *translation by* $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ of the plane spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.

Math 1553 Worksheet §2.6, 2.7, 2.9, 3.1

Solutions

1. Circle **TRUE** if the statement is always true, and circle **FALSE** otherwise.

a) If A is a 3×100 matrix of rank 2, then $\dim(\mathrm{Nul}A)=97$.

TRUE **FALSE**

b) If A is an $m\times n$ matrix and $Ax=0$ has only the trivial solution, then the columns of A form a basis for $\mathbf{R}^m$.

TRUE **FALSE**

c) The set $V=\left\{\begin{pmatrix}x\\y\\z\\w\end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x-4z=0\right\}$ is a subspace of $\mathbf{R}^4$.

TRUE **FALSE**

Solution.

a) False. By the Rank Theorem, $\mathrm{rank}(A)+\dim(\mathrm{Nul}A)=100$, so $\dim(\mathrm{Nul}A)=98$.

b) False. For example, $A=\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}$ has only the trivial solution for $Ax=0$, but its column space is a 2-dimensional subspace of $\mathbf{R}^3$.

c) True. V is $\mathrm{Nul}(A)$ for the 1×4 matrix A below, and therefore is automatically a subspace of $\mathbf{R}^4$:

$$A=\begin{pmatrix}1&0&-4&0\end{pmatrix}.$$

Alternatively, we could verify the subspace properties directly if we wished, but this is much more work!

(1) The zero vector is in V, since $0-4(0)0=0$.

(2) Let $u=\begin{pmatrix}x_1\\y_1\\z_1\\w_1\end{pmatrix}$ and $v=\begin{pmatrix}x_2\\y_2\\z_2\\w_2\end{pmatrix}$ be in V, so $x_1-4z_1=0$ and $x_2-4z_2=0$. We compute

$$u+v=\begin{pmatrix}x_1+x_2\\y_1+y_2\\z_1+z_2\\w_1+w_2\end{pmatrix}.$$

Is $(x_1+x_2)-4(z_1+z_2)=0$? Yes, since

$$(x_1+x_2)-4(z_1+z_2)=(x_1-4z_1)\;+\;(x_2-4z_2)=0+0=0.$$

(3) If $u = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ is in V then so is cu for any scalar c:

$$cu = \begin{pmatrix} cx \\ cy \\ cz \\ cw \end{pmatrix} \quad \text{and} \quad cx - 4cz = c(x-4z) = c(0) = 0.$$

2. Write a matrix A so that $\mathrm{Col} A = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\right\}$ and $\mathrm{Nul} A$ is the xz-plane.

Solution.

Many examples are possible. We'd like to design an A with the prescribed column span, so that $\left(A \mid 0\right)$ will have free variables x_1 and x_3. One way to do this is simply to leave the x_1 and x_3 columns blank, and make the second column $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$. This guarantees that A destroys the xz-plane and has the column span required.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

An alternative method for finding the same matrix: Write $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$. We want the column span to be the span of $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ and we want

$$A\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = xv_1 + zv_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for all } x \text{ and } z.$$

One way to do this is choose $v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$.

3. Let $A = \begin{pmatrix} 1 & -5 & -2 & -4 \\ 2 & 3 & 9 & 5 \\ 1 & 1 & 4 & 2 \end{pmatrix}$, and let T be the matrix transformation associated to A, so $T(x) = Ax$.

a) What is the domain of T? What is the codomain of T? Give an example of a vector in the range of T.

b) The RREF of A is $\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Is there a vector in the codomain of T which is not in the range of T? Justify your answer.

Solution.

a) The domain is $\mathbf{R}^4$; the codomain is $\mathbf{R}^3$. The vector $0 = T(0)$ is contained in the range, as is

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

b) Yes. The range of T is the column span of A, and from the RREF of A we know A only has two pivots, so its column span is a 2-dimensional subspace of $\mathbf{R}^3$. Since $\dim(\mathbf{R}^3) = 3$, the range is not equal to $\mathbf{R}^3$.

Math 1553 Worksheet §3.2, 3.3

Solutions

1. Which of the following transformations T are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the transformation is not one-to-one, find two vectors with the same image.

a) Counterclockwise rotation by 32° in $\mathbf{R}^2$.

b) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (z, x)$.

c) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (0, x)$.

d) The matrix transformation with standard matrix $A = \begin{pmatrix} 1 & 6 \\ -1 & 2 \\ 2 & -1 \end{pmatrix}$.

e) The matrix transformation with standard matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Solution.

a) This is both one-to-one and onto. If v is any vector in $\mathbf{R}^2$, then there is one and only one vector w such that $T(w) = v$, namely, the vector that is rotated −32° from v.

b) This is onto. If (a, b) is any vector in the codomain $\mathbf{R}^2$, then $(a, b) = T(b, 0, a)$, so (a, b) is in the range. It is not one-to-one though: indeed, $T(0, 0, 0) = (0, 0) = T(0, 1, 0)$. Alternatively, we could have observed that T is a matrix transformation and examined its matrix A: $T(x) = Ax$ for

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since A has a pivot in every row but not every column, T is onto but not one-to-one.

c) This is not onto. There is no (x, y, z) such that $T(x, y, z) = (1, 0)$. It is not one-to-one: for instance, $T(0, 0, 0) = (0, 0) = T(0, 1, 0)$.

d) The transformation T with matrix A is onto if and only if A has a pivot in every *row*, and it is one-to-one if and only if A has a pivot in every *column*. So we row reduce:

$$A = \begin{pmatrix} 1 & 6 \\ -1 & 2 \\ 2 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This has a pivot in every column, so T is one-to-one. It does not have a pivot in every row, so it is not onto. To find a specific vector b in $\mathbf{R}^3$ which is not in the image of T, we have to find a $b = (b_1, b_2, b_3)$ such that the matrix equation

$Ax = b$ is inconsistent. We row reduce again:

$$\left(A \mid b\right) = \left(\begin{array}{cc|c} 1 & 6 & b_1 \\ -1 & 2 & b_2 \\ 2 & -1 & b_3 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & \text{don't care} \\ 0 & 1 & \text{don't care} \\ 0 & 0 & -3b_1 + 13b_2 + 8b_3 \end{array}\right).$$

Hence any b_1, b_2, b_3 for which $-3b_1 + 13b_2 + 8b_3 \neq 0$ will make the equation $Ax = b$ inconsistent. For instance, $b = (1, 0, 0)$ is not in the range of T.

e) This matrix is already row reduced. We can see that does not have a pivot in every row *or* in every column, so it is neither onto nor one-to-one. In fact, if $T(x) = Ax$ then

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ x_3 \\ 0 \end{pmatrix},$$

so we can see that $(0, 0, 1)$ is not in the range of T, and that

$$T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = T\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

2. On your computer, go to the Interactive Transformation Challenge! Complete the Zoom, Reflect, and Scale challenges. If you complete a challenge in the optimal number of steps, the interactive demo will congratulate you. See if you can complete each of these challenges in the optimal number of steps.

3. The second little pig has decided to build his house out of sticks. His house is shaped like a pyramid with a triangular base that has vertices at the points $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(1, 1, 1)$.

The big bad wolf finds the pig's house and blows it down so that the house is rotated by an angle of 45° in a counterclockwise direction about the z-axis (look downward onto the xy-plane the way we usually picture the plane as $\mathbf{R}^2$), and then projected onto the xy-plane. Find the standard matrix A for the transformation T caused by the wolf.

Solution.

First notice that the little pig is a red herring, as it were—this is a question about the linear transformation T described in the last two lines.

To compute the matrix for T, we have to compute $T(e_1), T(e_2)$, and $T(e_3)$. To see the picture, let's put ourselves above the xy-plane (with the usual orientation of the x and y axes in the xy-plane), looking downward. For e_1 and e_2, it is as if we are applying $\begin{pmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then putting a zero in the z-coordinate each time. We find

$$T(e_1) = T\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} \qquad T(e_2) = T\begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}.$$

Rotating e_3 around the z-axis does nothing, and projecting onto the xy-plane sends it to zero, so $T(e_3) = 0$. Therefore, the matrix for T is

$$A = \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1 & 0\\1 & 1 & 0\\0 & 0 & 0\end{pmatrix}.$$

Math 1553 Worksheet §§3.4-3.6

Solutions

1. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) If A is a 3×4 matrix and B is a 4×2 matrix, then the linear transformation Z defined by $Z(x)=ABx$ has domain $\mathbf{R}^3$ and codomain $\mathbf{R}^2$.

b) If A is an $n\times n$ matrix and the equation $Ax=b$ has at least one solution for each b in $\mathbf{R}^n$, then the solution is *unique* for each b in $\mathbf{R}^n$.

c) Suppose A is an $n\times n$ matrix and every vector in $\mathbf{R}^n$ can be written as a linear combination of the columns of A. Then A must be invertible.

Solution.

a) False. In order for Bx to make sense, x must be in $\mathbf{R}^2$, and so Bx is in $\mathbf{R}^4$ and $A(Bx)$ is in $\mathbf{R}^3$. Therefore, the domain of Z is $\mathbf{R}^2$ and the codomain of Z is $\mathbf{R}^3$.

b) True. The first part says the transformation $T(x)=Ax$ is onto. Since A is $n\times n$, this is the same as saying A is invertible, so T is one-to-one and onto. Therefore, the equation $Ax=b$ has exactly one solution for each b in $\mathbf{R}^n$.

c) True. If the columns of A span $\mathbf{R}^n$, then A is invertible by the Invertible Matrix Theorem. We can also see this directly without quoting the IMT:

If the columns of A span $\mathbf{R}^n$, then A has n pivots, so A has a pivot in each row and column, hence its matrix transformation $T(x)=Ax$ is one-to-one and onto and thus invertible. Therefore, A is invertible.

2. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be rotation *clockwise* by $60°$. Let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation satisfying $U(1,0) = (-2,1)$ and $U(0,1) = (1,0)$.

a) Find the standard matrix for the composition $U \circ T$ using matrix multiplication.

b) Find the standard matrix for the composition $T \circ U$ using matrix multiplication.

c) Is rotating clockwise by $60°$ and then performing U, the same as first performing U and then rotating clockwise by $60°$?

Solution.

a) The matrix for T is $\begin{pmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$.

The matrix for U is $\begin{pmatrix} U(e_1) & U(e_2) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$.

The matrix for $U \circ T$ is

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1-\frac{\sqrt{3}}{2} & \frac{1}{2}-\sqrt{3} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

b) The matrix for $T \circ U$ is

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1+\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2}+\sqrt{3} & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

c) No. In (a) and (b), we found that the standard matrices for $U \circ T$ and $T \circ U$ are different, so the transformations are different.

Math 1553 Worksheet: 4.1-4.3

1. Answer true if the statement is *always* true. Otherwise, answer false. Justify your answer.

Suppose v_1, v_2, and v_3 are vectors in $\mathbf{R}^3$ and the volume of the parallelepiped naturally formed by v_1, v_2, and v_3 is 10. Then $\text{Span}\{v_1, v_2, v_3\}$ is all of $\mathbf{R}^3$.

Solution.

True. Let $A = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$. We know $\det(A) = 10 \neq 0$, so A is invertible. Therefore, every vector in $\mathbf{R}^3$ can be written as a linear combination of v_1, v_2, v_3.

2. Find the volume of the parallelepiped naturally formed by $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ using a cofactor expansion.

Solution.

We expand along the first row:

$$\begin{aligned}
\det\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & 1 \end{pmatrix} &= 2\det\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} - 1\det\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} + 1\det\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \\
&= 2(2-3) - 1(1+6) + 1(1+4) \\
&= -2 - 7 + 5 = -4.
\end{aligned}$$

The volume is $|-4| = 4$.

3. Let $A = \begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix}$.

a) Compute $\det(A)$ using row reduction.

b) Compute $\det(A^{-1})$ without doing any more work.

c) Compute $\det((A^T)^5)$ without doing any more work.

Solution.

a) Below, r counts the row swaps and s measures the scaling factors.

$$\begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} \xrightarrow{R_1 = \frac{R_1}{2}} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} \quad (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow[R_3 = R_3 + 3R_1,\ R_4 = R_4 - R_1]{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{pmatrix} \quad (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow{R_3 = R_3 + 4R_2} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{pmatrix} \quad (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow{R_4 = R_4 - \frac{R_2}{2}} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (r = 0,\ s = \frac{1}{2})$$

$$\det(A) = (-1)^0 \frac{1 \cdot 3 \cdot (-6) \cdot 1}{1/2} = -36.$$

b) From our notes, we know $\det(A^{-1}) = \dfrac{1}{\det(A)} = -\dfrac{1}{36}$.

c) $\det(A^T) = \det(A) = -36$. By the multiplicative property of determinants, if B is any $n \times n$ matrix, then $\det(B^n) = (\det B)^n$, so

$$\det((A^T)^5) = (\det A^T)^5 = (-36)^5 = -60,466,176$$

4. Play matrix tic-tac-toe!

Instead of X against O, we have 1 against 0. The 1-player wins if the final matrix has nonzero determinant, while the 0-player wins if the determinant is zero. You can change who goes first, and you can also modify the size of the matrix.

Click the link above, or copy and paste the url below:

`http://textbooks.math.gatech.edu/ila/demos/tictactoe/tictactoe.html`

Can you think of a winning strategy for the 0 player who goes first in the 2×2 case?
Is there a winning strategy for the 1 player if they go first in the 2×2 case?

Math 1553 Worksheet §5.1, 5.2

1. True or false. If the statement is always true, answer true and justify why it is true. Otherwise, answer false and give an example that shows it is false. In every case, assume that A is an $n \times n$ matrix.

a) To find the eigenvectors of A, we reduce the matrix A to row echelon form.

b) If v_1 and v_2 are linearly independent eigenvectors of A, then they must correspond to different eigenvalues.

Solution.

a) False. The RREF of A gives us almost no info about eigenvalues or eigenvectors. To get the eigenvectors corresponding to an eigenvalue λ, we put $A - \lambda I$ into RREF and write the solutions of $\left(A - \lambda I \mid 0\right)$ in parametric vector form.

b) False. For example, if $A = I_2$ then e_1 and e_2 are linearly independent eigenvectors both corresponding to the eigenvalue $\lambda = 1$.

2. In what follows, T is a linear transformation with matrix A. Find the eigenvectors and eigenvalues of A without doing any matrix calculations. (Draw a picture!)

a) $T =$ projection onto the xz-plane in $\mathbf{R}^3$.

b) $T =$ reflection over $y = 2x$ in $\mathbf{R}^2$.

Solution.

a) $T(x, y, z) = (x, 0, z)$, so T fixes every vector in the xz-plane and destroys every vector of the form $(0, a, 0)$ with a real. Therefore, $\lambda = 1$ and $\lambda = 0$ are eigenvalues and in fact they are the only eigenvalues since their combined eigenvectors span all of $\mathbf{R}^3$. The eigenvectors for $\lambda = 1$ are all vectors of the form $\begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$ where at least one of x and z is nonzero, and the eigenvectors for $\lambda = 0$ are all vectors of the form $\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$ where $y \neq 0$. In other words:

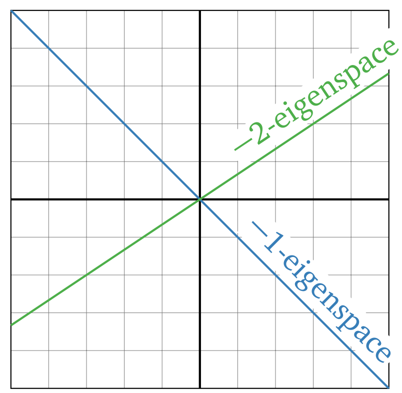
The 1-eigenspace consists of all vectors in $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$, while the 0-eigenspace consists of all vectors in $\mathrm{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$.

b) T fixes every vector along the line $y = 2x$, so $\lambda = 1$ is an eigenvalue and its eigenvectors are all vectors $\begin{pmatrix} t \\ 2t \end{pmatrix}$ where $t \neq 0$.

T flips every vector along the line perpendicular to $y = 2x$, which is $y = -\frac{1}{2}x$ (for example, $T(-2,1) = (2,-1)$). Therefore, $\lambda = -1$ is an eigenvalue and its eigenvectors are all vectors of the form $\begin{pmatrix} s \\ -\frac{1}{2}s \end{pmatrix}$ where $s \neq 0$.

3. Consider the matrix
$$A = -\frac{1}{5}\begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}.$$
Find, draw, and label the eigenspaces of A.

To save time, you may use the fact that the characteristic polynomial of A is
$$\det(A - \lambda I) = (\lambda + 2)(\lambda + 1).$$



Solution.

We were given the characteristic polynomial to save us time, but we could have computed it directly to find its roots:
$$0 = \det(A - \lambda I) = \left(-\frac{8}{5} - \lambda\right)\left(-\frac{7}{5} - \lambda\right) - \left(-\frac{2}{3}\right)\left(-\frac{3}{5}\right) = \frac{56}{25} + 3\lambda + \lambda^2 - \frac{6}{25}$$
$$= \lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1), \quad \text{so the eigenvalues are} \quad \lambda = -2, \quad \lambda = -1.$$
$$\left(A + 2I \mid 0\right) = \left(\begin{array}{cc|c} \frac{2}{5} & -\frac{3}{5} & 0 \\ -\frac{2}{5} & \frac{3}{5} & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right); \text{ (−2)-eigensp. has basis } \left\{\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}\right\}.$$
$$\left(A + I \mid 0\right) = \left(\begin{array}{cc|c} -\frac{3}{5} & -\frac{3}{5} & 0 \\ -\frac{2}{5} & -\frac{2}{5} & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right); \text{ (−1)-eigensp. has basis } \left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}.$$

Math 1553 Worksheet §5.4, 5.5

1. Answer yes, no, or maybe. Justify your answers. In each case, A is a matrix whose entries are real numbers.

a) If A is a 3×3 matrix with characteristic polynomial $-\lambda(\lambda-5)^2$, then the 5-eigenspace is 2-dimensional.

b) If A is an invertible 2×2 matrix, then A is diagonalizable.

c) A 3×3 matrix A can have a non-real complex eigenvalue with multiplicity 2.

Solution.

a) Maybe. The geometric multiplicity of $\lambda = 5$ can be 1 or 2. For example, the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 2-dimensional, whereas the matrix $\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 1-dimensional. Both matrices have characteristic polynomial $-\lambda(5-\lambda)^2$.

b) Maybe. The identity matrix is invertible and diagonalizable, but the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible but not diagonalizable.

c) No. If c is a (non-real) complex eigenvalue with multiplicity 2, then its conjugate $\bar{c}$ is an eigenvalue with multiplicity 2 since complex eigenvalues always occur in conjugate pairs. This would mean A has a characteristic polynomial of degree 4 or more, which is impossible for a 3×3 matrix.

2. $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$.

a) Find the eigenvalues of A, and find a basis for each eigenspace.

b) Is A diagonalizable? If your answer is yes, find a diagonal matrix D and an invertible matrix C so that $A = CDC^{-1}$. If your answer is no, justify why A is not diagonalizable.

Solution.

a) We solve $0 = \det(A - \lambda I)$.

$$0 = \det\begin{pmatrix} 2-\lambda & 3 & 1 \\ 3 & 2-\lambda & 4 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (-1-\lambda)(-1)^6 \det\begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = (-1-\lambda)((2-\lambda)^2 - 9)$$

$$= (-1-\lambda)(\lambda^2 - 4\lambda - 5) = -(\lambda+1)^2(\lambda - 5).$$

So $\lambda = -1$ and $\lambda = 5$ are the eigenvalues.

$\underline{\lambda = -1}$: $\left(A + I \mid 0\right) = \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 - R_1} \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1/3]{R_1 = R_1 - R_2}$

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$, with solution $x_1 = -x_2$, $x_2 = x_2$, $x_3 = 0$. The (-1)-eigenspace has basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

$\underline{\lambda = 5}$:

$$\left(A - 5I \mid 0\right) = \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 3 & -3 & 4 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right) \xrightarrow[R_3 = R_3/(-6)]{R_2 = R_2 + R_1} \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\text{then } R_2 \leftrightarrow R_3,\ R_1/(-3)]{R_1 = R_1 - R_3,\ R_2 = R_2 - 5R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

with solution $x_1 = x_2$, $x_2 = x_2$, $x_3 = 0$. The 5-eigenspace has basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

b) A is a 3×3 matrix that only admits 2 linearly independent eigenvectors, so A is not diagonalizable.

Math 1553 Worksheet §§5.6-6.3

1. Courage Soda and Dexter Soda compete for a market of 210 customers who drink soda each day. Today, Courage has 80 customers and Dexter has 130 customers. Each day:

70% of Courage Soda's customers keep drinking Courage Soda, while 30% switch to Dexter Soda.

40% of Dexter Soda's customers keep drinking Dexter Soda, while 60% switch to Courage Soda.

a) Write a stochastic matrix A and a vector x so that Ax will give the number of customers for Courage Soda and Dexter Soda (in that order) tomorrow.
You do not need to compute Ax.

$$A = \begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{pmatrix} \text{ and } x = \begin{pmatrix} 80 \\ 130 \end{pmatrix}.$$

b) A quick computation shows that the 1-eigenspace for this positive stochastic matrix A is spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Find the steady-state vector for A. In the long run, roughly how many daily customers will Courage Soda have?

The steady state vector is $w = \dfrac{1}{2+1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$.

As n gets large, $A^n \begin{pmatrix} 80 \\ 130 \end{pmatrix}$ approaches $210\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 140 \\ 70 \end{pmatrix}$. Courage will have roughly 140 customers.

2. Let W be the set of all vectors in $\mathbf{R}^3$ of the form $(x, x-y, y)$ where x and y are real numbers.

a) Find a basis for $W^\perp$.

b) Find the matrix B for orthogonal projection onto W.

Solution.

a) A vector in W has the form

$$\begin{pmatrix} x \\ x-y \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -y \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \text{so } W \text{ has basis } \left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}.$$

To get $W^\perp$ we find $\mathrm{Nul}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ which gives us

$$x_1 = -x_3, \quad x_2 = x_3, \quad x_3 = x_3 \text{ (free)},$$

so $W^\perp$ has basis $\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$.

b) Let A be the matrix whose columns are the basis vectors for W: $A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$.

We calculate $A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, so

$$B = A(A^T A)^{-1}A^T = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

MATH 1553, JANKOWSKI
MIDTERM 1, FALL 2019

Name		**Section**	

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered the text.
- Good luck!

This page was intentionally left blank.

Problem 1. [10 points]

These problems are true or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**. You do not need to justify your answer.

a) **T** **F** The augmented matrix below is in RREF.

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

b) **T** **F** If the RREF of an augmented matrix has a row of zeros, then the corresponding linear system of equations either has no solutions or infinitely many solutions.

c) **T** **F** Suppose $v_1,\ v_2,\ v_3,\ b$ are vectors in $\mathbf{R}^3$. If the vector equation

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = b$$

is inconsistent, then the vector equation $x_1 v_1 + x_2 v_2 = b$ must also be inconsistent.

d) **T** **F** If A is a 2×3 matrix and the solution set to $Ax = 0$ is a plane in $\mathbf{R}^3$, then the equation $Ax = b$ must be inconsistent for some b in $\mathbf{R}^2$.

e) **T** **F** If A is a 2×2 matrix and the equation $Ax = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$ is consistent, then the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ must be in the span of the columns of A.

Solution.

a) True.

b) False. For example, the augmented system $\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$ has a unique solution.

c) True. If b is not in $\text{Span}\{v_1, v_2, v_3\}$ then it cannot be in $\text{Span}\{v_1, v_2\}$, every vector in $\text{Span}\{v_1, v_2\}$ is contained in $\text{Span}\{v_1, v_2, v_3\}$.

d) True. Since A is 2×3 and the solution set to $Ax = 0$ is a plane, we have two free variables and thus A only has one pivot, so it cannot have a pivot in every row and thus $Ax = b$ will be inconsistent for some b in $\mathbf{R}^2$.

e) True: the column span of A contains all scalar multiples of $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, so it includes $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Extra space for scratch work on problem 1

Problem 2. [11 points]

You don't need to show work on (b). Parts (a)-(c) are 2 points each, and part (d) is 5 points.

a) Complete the following definition (be mathematically precise!):
Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. We say $\{v_1, v_2, \ldots, v_p\}$ is *linearly independent* if...

the equation $x_1v_1 + \cdots + x_pv_p = 0$ has only the trivial solution

$$x_1 = \cdots = x_p = 0.$$

b) Suppose a homogeneous system of 4 linear equations in 3 unknowns corresponds to an augmented matrix with exactly two pivots. Then the solution set for the system is a:

(circle one answer) **point** $\boxed{\textbf{line}}$ **plane**

in:

(circle one answer) $\mathbf{R}$ $\mathbf{R}^2$ $\boxed{\mathbf{R}^3}$ $\mathbf{R}^4$.

c) Is there a 2×2 matrix A so that the solution set for the equation $Ax = 0$ is the line $x_1 = x_2 + 1$? If yes, write such an A. If no, justify why there is no such A.

No. The solution set to $Ax = 0$ must include the origin $x_1 = x_2 = 0$, which is not on the line $x_1 = x_2 + 1$.

d) Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ -2 \\ c \end{pmatrix}$.

Find all values of c (if there are any) so that v_3 is a linear combination of v_1 and v_2.

We solve the equation $x_1v_1 + x_2v_2 = v_3$.

$$\left(\begin{array}{cc|c} 1 & -2 & 2 \\ 2 & -1 & -2 \\ -1 & 3 & c \end{array}\right) \xrightarrow[R_3=R_3+R_1]{R_2=R_2-2R_1} \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 3 & -6 \\ 0 & 1 & c+2 \end{array}\right) \xrightarrow[\text{then } R_3=R_3-R_2]{R_3=R_3/3} \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & c+4 \end{array}\right).$$

This is consistent if and only if $c + 4 = 0$, so $\boxed{c = -4}$.

Extra space for work on problem 2

Problem 3.

Parts (a) and (b) are unrelated. Part (a) is worth 5 points. Part (b) is worth 7 points.

a) Councilman Jamm loves the linear system of equations

$$2x - hy = k$$
$$4x + 10y = 5,$$

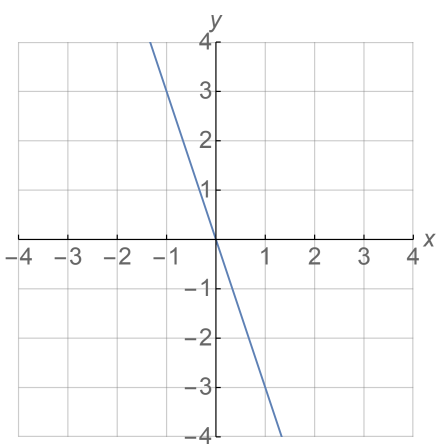
where h and k are real numbers. Find all values of h and k (if there are any) so that the system has infinitely many solutions.

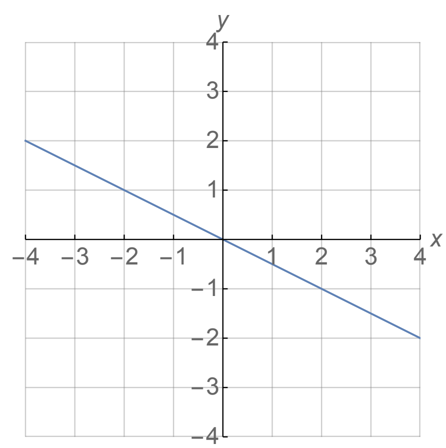
Solution: $\left(\begin{array}{cc|c} 2 & -h & k \\ 4 & 10 & 5 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 2 & -h & k \\ 0 & 10+2h & 5-2k \end{array}\right)$.

The augmented matrix will have exactly one pivot (and no pivot in the final column) precisely when $10 + 2h = 0$ but $5 - 2k = 0$, so $h = -5$ and $k = \frac{5}{2}$.

b) Let $A = \begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$. On the left graph, draw the span of the columns of A. On the right graph, draw the solution set for the equation $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The column span is Span$\left\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right\}$ which is the line $y = -3x$. For the homogeneous solution set, $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ -3 & -6 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 + 3R_1} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ so $x + 2y = 0$ yielding the line $y = -x/2$ which is Span$\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$.





Extra space for work on problem 3

Problem 4. [10 points]

Consider the following linear system of equations in the variables $x_1,\ x_2,\ x_3,\ x_4$:

$$\begin{aligned} x_1 - 2x_2 + x_4 &= 1 \\ x_1 - 2x_2 + x_3 + x_4 &= -2 \\ 3x_1 - 6x_2 + 2x_3 + 3x_4 &= -3. \end{aligned}$$

a) Write the augmented matrix corresponding to this system, and put the augmented matrix into RREF.

b) The system is consistent. Write the set of solutions to the system of equations in parametric vector form.

c) Write *one* vector that is not the zero vector and that is a solution for the corresponding homogeneous system of equations below. You do not need to show your work for this part.

$$\begin{aligned} x_1 - 2x_2 + x_4 &= 0 \\ x_1 - 2x_2 + x_3 + x_4 &= 0 \\ 3x_1 - 6x_2 + 2x_3 + 3x_4 &= 0. \end{aligned}$$

Solution.

a)

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 1 \\ 1 & -2 & 1 & 1 & -2 \\ 3 & -6 & 2 & 3 & -3 \end{array}\right) \xrightarrow[R_3=R_3-2R_1]{R_2=R_2-R_1} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 & -6 \end{array}\right) \xrightarrow{R_3=R_3-2R_2} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

b) The RREF in part (a) shows that x_2 and x_4 are free and

$$x_1 = 1 + 2x_2 - x_4 \qquad x_2 = x_2 \qquad x_3 = -3 \qquad x_4 = x_4.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 + 2x_2 - x_4 \\ x_2 \\ -3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

c) Any vector that lives in Span $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a homogeneous solution, so we just need to pick a nonzero vector there. For example $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ are correct answers.

Extra space for work on problem 4

Problem 5. [7 points]

Parts (a) and (b) are unrelated.

a) Write a single matrix A that satisfies all of the following conditions:

- The equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is consistent, and the solution set is a line.
- The equation $Ax = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is inconsistent.

Many answers possible. We need $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ in the column span and exactly one free variable, and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ not in the column span. For example

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

b) Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbf{R}^4$. Which of the following statements must be true? Circle all that apply.

(i) If $\{v_1, v_2, v_3, v_4\}$ is linearly dependent, then v_4 is in $\mathrm{Span}\{v_1, v_2, v_3\}$.

$\boxed{\text{(ii)}}$ If $\{v_1, v_2, v_3\}$ is linearly independent, then $\{v_1, v_2\}$ is linearly independent.

$\boxed{\text{(iii)}}$ If $\{v_1, v_2, v_3, v_4\}$ is linearly independent, then $\mathrm{Span}\{v_1, v_2, v_3, v_4\} = \mathbf{R}^4$.

Extra space for work on problem 5

MATH 1553, JANKOWSKI
MIDTERM 2, FALL 2019

Name		Section	

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As usual: RREF means "reduced row echelon form."
- As usual: $e_1, e_2, \ldots, e_n$ denote the standard unit coordinate vectors in $\mathbf{R}^n$.
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page. use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered the text.
- Good luck!

This page was intentionally left blank.

Problem 1.

[10 points]

These problems are true or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**. You do not need to justify your answer, and there is no partial credit.

a) **T** **F** Suppose A is an $n \times n$ matrix and the equation $Ax = b$ has infinitely many solutions for some b in $\mathbf{R}^n$. Then A is not invertible.

b) **T** **F** If A is an $n \times n$ matrix and the transformation given by $T(x) = Ax$ is invertible, then $\mathrm{Nul}(A) = \mathrm{Col}(A)$.

c) **T** **F** The set $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - 3y + 2z + w = 0 \right\}$ is a 3-dimensional subspace of $\mathbf{R}^4$.

d) **T** **F** Suppose A is an $m \times n$ matrix and $m < n$. Then the matrix transformation $T(x) = Ax$ is not onto.

e) **T** **F** If A is a $k \times 5$ matrix and the columns of A form a basis for $\mathbf{R}^k$, then $k = 5$.

Solution.

a) True, by the Invertible Matrix Theorem.

b) False. If A is invertible then $\mathrm{Nul}(A)$ only contains the zero vector.

c) True. It is $\mathrm{Nul}\begin{pmatrix} 1 & -3 & 2 & 1 \end{pmatrix}$ which is a subspace of $\mathbf{R}^4$ and gives 3 free variables in the corresponding homogeneous system.

d) False. It might be onto, for example when A is the 2×3 matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

e) True. Since A is $k \times 5$ we know A has 5 columns. If the 5 columns form a basis for $\mathbf{R}^k$ then they are linearly independent and span $\mathbf{R}^k$, so $\dim(\mathbf{R}^k) = 5$ hence $k = 5$.

Extra space for scratch work on problem 1

Problem 2.

[12 points]

Short answer. Show your work in part (a).

a) Let $A = \begin{pmatrix} 1 & -3 \\ 2 & 3 \end{pmatrix}$. Find A^{-1}.

b) Suppose A is a 4×5 matrix. Which of the following statements must be true? Clearly circle all that apply.

(i) If $\dim(\mathrm{Nul}\ A) = 2$, then $\mathrm{Col}(A) = \mathbf{R}^3$.

(ii) The matrix transformation $T(x) = Ax$ is not one-to-one.

c) Suppose $T : \mathbf{R}^4 \to \mathbf{R}^3$ is an *onto* linear transformation with standard matrix A, so $T(x) = Ax$. Which of the following statements *must* be true? Clearly circle all that apply.

(i) The matrix A has exactly three pivot columns.

(ii) For each vector v in $\mathbf{R}^3$, there is at least one vector x in $\mathbf{R}^4$ so that $T(x) = v$.

(iii) For each vector x in $\mathbf{R}^4$, there is a vector v in $\mathbf{R}^3$ so that $T(x) = v$.

(iv) $\mathrm{Span}\left\{T(e_1),\ T(e_2),\ T(e_3),\ T(e_4)\right\} = \mathbf{R}^3$.

d) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x \geq y \right\}$. Which properties of a subspace does V satisfy? Clearly circle all that apply.

(i) V contains the zero vector.

(ii) V is closed under addition.

(iii) V is closed under scalar multiplication.

Solution to problem 2

(a) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then A is invertible if $ad - bc \neq 0$, and $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$$A^{-1} = \frac{1}{3-(-6)}\begin{pmatrix} 3 & 3 \\ -2 & 1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 3 & 3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ -2/9 & 1/9 \end{pmatrix}.$$

It is fine if the student leaves it in either of the last two forms.

(b) (i) is NOT true: Col A is a 3-dimensional subspace of $\mathbf{R}^5$, it is NOT equal to $\mathbf{R}^3$.
(ii) is true, since A cannot have a pivot in every column (5 columns, max of 4 pivots).

(c) All 4 are true! Note that (iii) is true of any transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$. Part (iv) is just the statement that the columns of A span $\mathbf{R}^3$, which is true since T is onto.

(d) (i) is true: V contains the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ since $0 \geq 0$.

(ii) is true: Suppose $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are in V. Since $x_1 \geq y_1$ and $x_2 \geq y_2$, we have

$$x_1 + x_2 \geq y_1 + y_2.$$

(iii) is not true: For example $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ is in V since $5 > 1$, but $\begin{pmatrix} -5 \\ -1 \end{pmatrix}$ is not in V since $-5 < -1$.

Geometrically, V is just the region of $\mathbf{R}^2$ that lies on and beneath the line $y = x$.

Problem 3. [10 points]

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation counterclockwise by 45°, and let $U : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation given by

$$U\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 2x_2 \\ x_2 \\ x_2 - x_1 \end{pmatrix}.$$

a) Write the standard matrix A for T. Simplify your answer (do not leave it in terms of sines and cosines).

$$A = \begin{pmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

b) Find the standard matrix B for U. $B = \begin{pmatrix} U(e_1) & U(e_2) \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$.

c) Circle the composition that makes sense: $T \circ U$ $\boxed{U \circ T}$.

d) (Unrelated to (a) through (c))
Let $R : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first performs reflection across the line $y = x$, then does reflection across the y-axis. Find the standard matrix C for R.

Write G for the matrix that reflects across $y = x$, so $G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Write H for the matrix that reflects across the y-axis, so $H = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

We want G followed by H, so $C = HG$.

$$C = HG = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Extra space for work on problem 3

Problem 4. [10 points]

Let $A = \begin{pmatrix} 1 & -1 & 2 & -2 \\ -2 & 2 & -4 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

a) Find a basis for the null space of A.

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -2 & 0 \\ -2 & 2 & -4 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_2=R_2+2R_1} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow[\text{then } R_1=R_1-2R_2]{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We see x_2 and x_4 are free, $x_3 = 0$, and $x_1 = x_2 + 2x_4$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 + 2x_4 \\ x_2 \\ 0 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

A basis for Nul A is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

b) Write a basis for the column space of A, and write dim(Col A). You do not need to show your work on this part.

The pivot columns form a basis for Col A, so a basis is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \right\}.$$

Other answers are possible. Since a basis for Col A has 2 vectors, we know dim(Col A) $= 2$.

c) Find a vector b in $\mathbf{R}^3$ which is not in Col A. Briefly show your work.

Many answers possible. Any vector in $\mathbf{R}^3$ that does not have its second entry as (-2 times its first) will fail to be in Col$A$, for example

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 25 \\ 1 \\ 7 \end{pmatrix}.$$

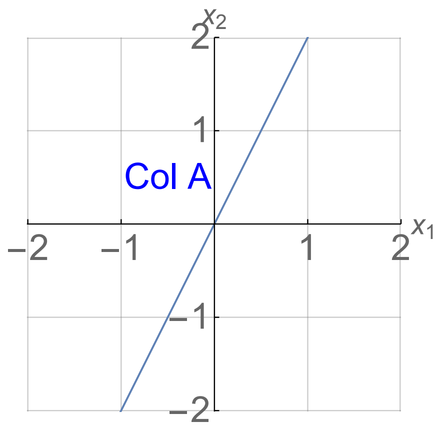
Extra space for work on problem 4

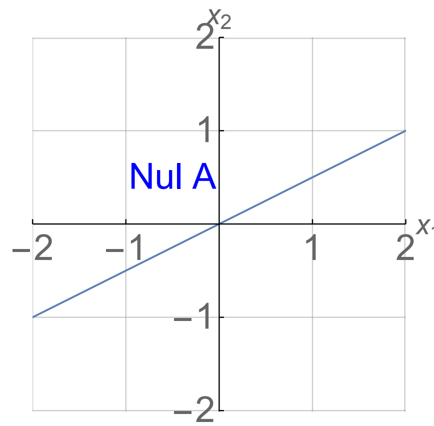
Problem 5.

[8 points]

Problems (a) and (b) are unrelated.

a) Find the matrix A so that Col A and Nul A are given below.





We need Col $A = \text{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ and Nul$A$ to be the line $x_2 = \dfrac{x_1}{2}$, so $x_1 = 2x_2$. Many answers are possible. For example,

$$A = \begin{pmatrix}1 & -2\\ 2 & -4\end{pmatrix}.$$

b) Suppose $T : \mathbf{R}^3 \to \mathbf{R}^2$ is a linear transformation satisfying

$$T\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}.$$

Find $T\begin{pmatrix}4\\2\\-4\end{pmatrix}$.

$\begin{pmatrix}4\\2\\-4\end{pmatrix} = 4\begin{pmatrix}1\\0\\-1\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix}$, so by linearity:

$$T\begin{pmatrix}4\\2\\-4\end{pmatrix} = T\left(4\begin{pmatrix}1\\0\\-1\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = 4T\begin{pmatrix}1\\0\\-1\end{pmatrix} + 2T\begin{pmatrix}0\\1\\0\end{pmatrix} = 4\begin{pmatrix}1\\2\end{pmatrix} + 2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2\\10\end{pmatrix}.$$

Extra space for work on problem 5

MATH 1553, JANKOWSKI
MIDTERM 3, FALL 2019

Name		Section	

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- As always, $e_1, \ldots, e_n$ are the standard unit coordinate vectors in $\mathbf{R}^n$.
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page. use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered the text.
- Good luck!

This page was intentionally left blank.

Problem 1. [10 points]

These problems are true or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**. You do not need to justify your answer. Assume that all entries of matrices A and B are real numbers.

a) **T** **F** If A is a 3×3 matrix and its characteristic polynomial is $-\lambda^3 + 2\lambda^2 - 17\lambda$, then $\det(A) = 0$.

b) **T** **F** Suppose A is a 3×3 matrix with columns v_1, v_2, v_3. If $v_1 - v_2 + v_3 = 0$, then the determinant of A must be zero.

c) **T** **F** Suppose A and B are $n \times n$ matrices with the same characteristic polynomial. If A is diagonalizable, then B must also be diagonalizable.

d) **T** **F** Suppose A and B are $n \times n$ matrices. If $\det(A) = \det(B)$, then $\det(A-B) = 0$.

e) **T** **F** Suppose A is a 3×3 matrix and that v is an eigenvector of A corresponding to the eigenvalue $\lambda = -1$. Then $-2v$ must also be an eigenvector of A corresponding to the eigenvalue $\lambda = -1$.

Solution.

a) True. $\det(A) = \det(A - 0I) = -0^3 + 2 \cdot 0^2 - 17 \cdot 0 = 0$.

b) True. The columns of A are linearly dependent, so $\det(A) = 0$.

c) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

d) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ have determinant 1 but

$$\det(A - B) = \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4.$$

e) True. $A(-2v) = -2Av = -2(-v) = 2v$ so $A(-2v) = -(-2v)$. Alternatively, we could note that the (-1)-eigenspace is closed under scalar multiplication and $-2v \neq 0$, so $-2v$ is a (-1)-eigenvector.

Extra space for scratch work on problem 1

Problem 2.

[12 points]

Short answer. All parts are unrelated. You do not need to show your work. In each part, all entries of all matrices are real numbers.

a) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 2$. Fill in the blank:

$$\det\begin{pmatrix} a & b & c \\ g & h & i \\ -2d+3g & -2e+3h & -2f+3i \end{pmatrix} = \_\_\_\_\_\_\_\_\_\_.$$

b) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects across the line $y = -2x$, and let A be the standard matrix for T. Which of the following are true? Circle all that apply.

(i) The 1-eigenspace of A is $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$.

(ii) A is diagonalizable.

(iii) $\det(A + I) = 0$.

c) Write a 2×2 matrix A that is diagonalizable but not invertible.

d) Suppose A is a 3×3 matrix and its characteristic polynomial is

$$\det(A - \lambda I) = -(\lambda - 5)(\lambda - 3)^2.$$

Which of the following *must* be true? Circle all that apply.

(i) The 5-eigenspace of A has dimension 1.

(ii) The homogeneous system given by the equation $(A - 3I)x = 0$ has two free variables.

(iii) For each b in $\mathbf{R}^3$, the equation $Ax = b$ is consistent.

e) Is there a 3×3 matrix A with the property that the 2-eigenspace of A is $\mathbf{R}^3$? If your answer is yes, write such a matrix A.

Solution.

a) This matrix is obtained from the original by switching the last two rows (multiplying the determinant by -1), scaling the new last row by -2, and then adding 3 times row two to row three (no change). Thus the determinant is $2 \cdot -1 \cdot -2 = 4$.

b) All of them are true! We see (i) is true since T fixes all vectors along the line $y = -2x$, and (ii) is true because A is a 2×2 matrix with two distinct eigenvalues 1 and -1. Also, (iii) is true because -1 is an eigenvalue of A (recall that A flips all vectors perpendicular to the line $y = -2x$).

c) Many examples possible, for example $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

d) (i) and (iii) are true. We see (i) is true because 5 has algebraic multiplicity 1 thus has geometric multiplicity 1. For (ii), if $(A - 3I)x = 0$ has two free variables then the *geometric* mutliplicity of $\lambda = 3$ is 2, which is not necessarily true. For (iii), A is invertible since 0 is not an eigenvalue of A, so $Ax = b$ is guaranteed to be consistent no matter what b is.

e) Yes. There is exactly one such matrix, namely $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Problem 3.

[8 points]

Let $A = \begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}$.

a) Find the characteristic polynomial of A and the complex eigenvalues of A. Simplify your eigenvalues completely.

Solution:

$$\det(A - \lambda I) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 6\lambda + 13.$$

The eigenvalues are

$$\lambda = \frac{6 \pm \sqrt{6^2 - 4(13)}}{2} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i.$$

b) For the eigenvalue of A with *negative* imaginary part, find a corresponding eigenvector v.

Solution: Let $\lambda = 3 - 2i$.

$$\left(A - \lambda I \mid 0\right) = \left(\begin{array}{cc|c} 2-(3-2i) & 5 & 0 \\ -1 & 4-(3-2i) & \end{array}\right) = \left(\begin{array}{cc|c} -1+2i & 5 & 0 \\ -1 & 1+2i & 0 \end{array}\right).$$

By the 2×2 eigenvector trick, we know that since $A - \lambda I$ is not invertible, then and eigenvector is $\begin{pmatrix} -b \\ a \end{pmatrix}$ where $(a\ b)$ is the first row of $A - \lambda I$. Therefore, an eigenvector for λ is $v = \begin{pmatrix} -5 \\ -1+2i \end{pmatrix}$. Any nonzero complex scalar multiple of that is also an eigenvector, so another correct answer is

$$v = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}.$$

Alternatively, $(A - \lambda I\ 0) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -1-2i & 0 \\ 0 & 0 & 0 \end{array}\right)$ which gives $v = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$.

c) Using your answer from (b), write an eigenvector w of A corresponding to the eigenvalue with *positive* imaginary part. You do not need to show your work on this part.

Solution: $w = \overline{v}$, so $w = \begin{pmatrix} -5 \\ -1-2i \end{pmatrix}$ (using first v above), or $w = \begin{pmatrix} 1-2i \\ 1 \end{pmatrix}$, etc.

Extra space for work on problem 3

Problem 4. [10 points]

Let $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}$.

a) Find the eigenvalues of A.
Solution:

$$\begin{aligned} \det(A-\lambda I) &= (-1-\lambda)\Big[(3-\lambda)(-2-\lambda)+4\Big] \\ &= -(\lambda+1)[\lambda^2-\lambda-6+4] = -(\lambda+1)[\lambda^2-\lambda-2] \\ &= -(\lambda+1)(\lambda+1)(\lambda-2) = -(\lambda+1)^2(\lambda-2) \end{aligned}$$

The eigenvalues are $\lambda = -1$ and $\lambda = 2$.

b) For each eigenvalue of A, find a basis for the corresponding eigenspace.
Solution: <u>For $\lambda = -1$</u>:

$$\big(A+I \mid 0\big) = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus $x_1 = x_1$ (free), $x_2 = \frac{x_3}{2}$, and $x_3 = x_3$ (free).

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_3/2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix}. \quad \text{Basis}: \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} \right\}.$$

<u>For $\lambda = 2$:</u>

$$\big(A-2I \mid 0\big) = \left(\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \quad \text{Basis}: \left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

c) Is A diagonalizable? If your answer is yes, write an invertible 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$. If your answer is no, justify why A is not diagonalizable.

Yes: $A = CDC^{-1}$ where $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Extra space for work on problem 4

Problem 5.

[10 points]

Parts (a) and (b) are unrelated.

a) Suppose A and B are 4×4 matrices satisfying

$$\det(A) = 5, \quad \det(AB^{-1}) = 10.$$

Find $\det(-2B)$. Simplify your answer completely.

Solution:

$$\det(A)\det(B^{-1}) = 10, \quad \det(B^{-1}) = \frac{10}{\det(A)} = \frac{10}{5} = 2, \qquad \det(B) = \frac{1}{2}.$$

Therefore, $\det(-2B) = (-2)^4 \det(B) = 16 \cdot \dfrac{1}{2} = 8$.

b) Let $A = C \begin{pmatrix} -1 & 0 \\ 0 & 1/2 \end{pmatrix} C^{-1}$, where the columns of C are (in order) $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

(i) Write one nonzero vector v so that $A^n v$ approaches the zero vector as n gets very large (you do not need to show your work on part (i)).

Solution: This problem uses our geometric interpretation of diagonalization. Since $\{v_1, v_2\}$ is a basis for $\mathbf{R}^2$, we can write $v = x_1 v_1 + x_2 v_2$ for some scalars x_1 and x_2. Since $Av_1 = -v_1$ and $Av_2 = \frac{1}{2}v_2$, we get

$$Av = A(x_1 v_1 + x_2 v_2) = x_1 A v_1 + x_2 A v_2 = -x_1 v_1 + \frac{1}{2} x_2 v_2,$$

$$A^2 v = A^2(x_1 v_1 + x_2 v_2) = A(-x_1 v_1 + \frac{1}{2} x_2 v_2) = (-1)^2 x_1 v_1 + \left(\frac{1}{2}\right)^2 x_2 v_2,$$

and in general

$$A^n v = A^n(x_1 v_1 + x_2 v_2) = (-1)^n x_1 v_1 + \left(\frac{1}{2}\right)^n x_2 v_2.$$

As n gets huge, $(-1)^n x_1 v_1$ just keeps flipping between $-x_1 v_1$ and $x_1 v_1$, whereas $\left(\frac{1}{2}\right)^n x_2 v_2$ approaches the origin.

Thus, as n gets large, $A^n v$ will approach the origin precisely when $v = 0 + x_2 v_2$ for some scalar x_2. We were asked for a nonzero v, so $\boxed{v = \begin{pmatrix} 1 \\ -3 \end{pmatrix}}$ or any nonzero scalar multiple of $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

The nonzero scalar multiples of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are the *opposite* of what we want, since cv_1 (where c is a nonzero scalar) just keeps flipping between $-cv_1$ and cv_1, never getting closer to the origin as n gets large.

(ii) Find $A^{10}\begin{pmatrix}1\\2\end{pmatrix}$. Show your work!

Solution: We saw in (i) that $A^n\begin{pmatrix}1\\2\end{pmatrix} = (-1)^n\begin{pmatrix}1\\2\end{pmatrix}$, so

$$A^{10}\begin{pmatrix}1\\2\end{pmatrix} = (-1)^{10}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}.$$

Many students tried to multiply the entire thing out by finding C^{-1} and doing $CD^{10}C^{-1}\begin{pmatrix}1\\2\end{pmatrix}$, but this was unnecessary and almost always led to an algebraic error or an unsimplified final answer.

(iii) Clearly circle your answer: Is A^4 diagonalizable? $\boxed{\text{YES}}$ NO
(no justification required for (iii))

Solution: Yes. $A^{10} = CD^{10}C^{-1}$ where D^{10} is the diagonal matrix

$$D^{10} = \begin{pmatrix}1 & 0\\ 0 & (\frac{1}{2})^{10}\end{pmatrix}.$$

Extra space for work on problem 5

MATH 1553
FINAL EXAM, FALL 2019

Name	

Circle the name of your instructor below:

Blekherman G1-G3 (11:15 AM - 12:05 PM) Blekherman H1-H3 (12:20 - 1:10 PM)

Bonetto L1-L5 Chen A5-A8 (8:00 - 8:50 AM) Chen C1-C6 (9:05 - 9:55 AM)

Cui E1-E3 Dieci E5-E8 Goldsztein J4-J8

Jankowski A0-A4 Pan J1-J3

DO NOT WRITE IN THE TABLE BELOW. It will be used to record scores.

1	2	3	4	5	6	7	8	9	10	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- You may not use any calculators or aids of any kind (cell phones, notes, text, etc.).
- Unless a problem specifies that no work is required, show your work or you may receive little or no credit, even if your answer is correct.
- If you run out of room on a page, you may use its back side to finish the problem, but please indicate this.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness. Good luck!
- Assume that the entries in all matrices are real numbers unless otherwise specified.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam.

Problem 1.

[1 point each]

True or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**. You do not need to show work or justify your answer. In every case, A is a matrix whose entries are real numbers.

a) **T** **F** If $\{v_1, v_2, v_3\}$ is a linearly independent set of vectors in $\mathbf{R}^4$, then

$$\{v_1 + v_3,\ v_2 + v_3,\ v_3\}$$

is also linearly independent.

b) **T** **F** If A is a 6×4 matrix, then the equation $Ax = 0$ must have a non-trivial solution.

c) **T** **F** Suppose A is a 4×4 matrix and $A^3 = I$. Then A is invertible.

d) **T** **F** If A is an invertible matrix and λ is an eigenvalue of A, then $\dfrac{1}{\lambda}$ is an eigenvalue of A^{-1}.

e) **T** **F** If $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is a 2×2 matrix which is not diagonalizable, then $a = d$.

f) **T** **F** If an $n \times n$ matrix A has two eigenvectors u and v corresponding to the same eigenvalue λ, then u must be a scalar multiple of v.

g) **T** **F** There are exactly two real values of c for which the distance between the vectors v and w in $\mathbf{R}^3$ written below is equal to 4.

$$v = \begin{pmatrix} 1 \\ -2c \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

h) **T** **F** Suppose W is a subspace of $\mathbf{R}^n$ and x is a vector in $\mathbf{R}^n$. If the orthogonal projection of x onto $W^\perp$ is the zero vector, then x is in W.

i) **T** **F** Suppose A is an invertible 3×3 matrix. Then the product of the second row of A and the third column of A^{-1} must equal 0.

j) **T** **F** If A is a 5×5 matrix, then A has at least one real eigenvalue with geometric multiplicity greater than or equal to 1.

Solution to problem 1, "Exam" version.

a) True.

b) False: If A has 4 pivots then $Ax = 0$ has only the trivial solution.

c) True: $A(A^2) = A^2(A) = I$, so $A^2 = A^{-1}$.

d) True: $Av = \lambda v$ for an eigenvector v of A gives

$$A^{-1}Av = A^{-1}\lambda v \qquad v = \lambda A^{-1}v \qquad \frac{1}{\lambda}v = A^{-1}v.$$

e) True: If $a \neq d$ then A is a 2×2 matrix with two distinct real eigenvalues and thus must be diagonalizable.

f) False: for example $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The vectors $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are both eigenvectors for $\lambda = 1$ but u is not a scalar multiple of v.

g) True. The system

$$\sqrt{(-1-1)^2 + (2+2c)^2 + (-1-1)^2} = 4$$

has exactly two solutions, $c = -1 \pm \sqrt{2}$.

h) True. $x = x_W + x_{W^\perp} = x_W + 0$, so $x = x_W$, hence x is in W.

i) True: the 23-entry of AA^{-1} is 0 since $AA^{-1} = I$.

j) True: A is 5×5 so it has at least one real eigenvalue.

Problem 2.

Short answer. You do not need to show any work.

a) (3 points total) Which of the following transformations are linear? Circle all that apply.

(i) $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (x, y + 1)$.

(ii) $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (2x - \sin(y), \pi x)$.

(iii) $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (x \ln(2), y)$.

b) (3 points total) Suppose $T : \mathbf{R}^m \to \mathbf{R}^n$ and $U : \mathbf{R}^n \to \mathbf{R}^m$ are linear transformations. Which of the following *must* be true? Clearly circle all that apply.

(i) The transformation $T \circ U$ has domain $\mathbf{R}^n$ and codomain $\mathbf{R}^n$.

(ii) If $m > n$, then $T \circ U$ is not onto.

(iii) U is one-to-one if for every x in $\mathbf{R}^n$, there is a y in $\mathbf{R}^m$ so that $U(x) = y$.

c) (4 points total) Let W be the subspace of $\mathbf{R}^3$ consisting of all (x, y, z) in $\mathbf{R}^3$ satisfying $x - 3y - z = 0$, and let B be the matrix for orthogonal projection onto W. Which of the following are true? Clearly circle all that apply.

"**Exam**" version

(i) $\text{rank}(B) = 1$.

(ii) $B \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$.

(iii) If v is a vector in $\mathbf{R}^3$ and v is not in W, then $Bv = 0$.

(iv) $(\text{Col } B)^{\perp}$ is a line in $\mathbf{R}^3$.

Problem 3.

Short answer. You do not need to justify your answer, and there is no partial credit. Each part of (a) is worth 2 points, each part of (b) is worth 1 point, and each part of (c) is worth 2 points.

a) (4 points total)
Which of the following matrices are diagonalizable? Clearly circle all that apply.

$\boxed{\text{(i)}}$ $A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$

(ii) $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 2 \end{pmatrix}$

b) (2 points total) Consider the matrix A below, where h is a real number.

$$A = \begin{pmatrix} 2 & h & 0 \\ 0 & 1 & h \\ 0 & 0 & 1 \end{pmatrix}.$$

Which of the following statements must be true? Clearly circle all that apply.

$\boxed{\text{(i)}}$ A must be invertible, no matter what h is.

(ii) If $h = 1$, then $\lambda = 1$ is an eigenvalue of A with geometric multiplicity 2.

c) (4 points total) "**Exam**" version
Suppose A and B are 2×2 matrices satisfying

$$\det(A) = -2, \qquad \det(B) = 5.$$

(i) Fill in the blank: $\det(A^2B^{-1}) =$ $\underline{\quad \frac{4}{5} \quad}$.

(ii) Fill in the blank: $\det(6A) =$ $\underline{\quad -72 \quad}$.

Problem 4.

Short answer. On this page, **show your work**!

a) **“Exam”** version

Suppose that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation and

$$T\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}6\\4\end{pmatrix}, \qquad T\begin{pmatrix}4\\0\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}.$$

(i) (3 points) Find $T\begin{pmatrix}2\\1\end{pmatrix}$. Enter your answer here: $T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$.

$$\begin{pmatrix}2\\1\end{pmatrix} = \frac{1}{3}\left[\begin{pmatrix}2\\3\end{pmatrix} + \begin{pmatrix}3\\-1\end{pmatrix}\right],$$

so by linearity of T,

$$T\begin{pmatrix}2\\1\end{pmatrix} = \frac{1}{3}\left[T\begin{pmatrix}2\\3\end{pmatrix} + T\begin{pmatrix}3\\-1\end{pmatrix}\right] = \frac{1}{3}\left[\begin{pmatrix}6\\4\end{pmatrix} + \begin{pmatrix}3\\-1\end{pmatrix}\right] = \begin{pmatrix}3\\1\end{pmatrix}.$$

(ii) (1 point) Clearly circle your answer: Is T one-to-one? $\boxed{\text{YES}}$ NO

(no work required for part (ii))

b) Let $A = \begin{pmatrix}-1 & 2\\ 1 & 1\\ 2 & -1\end{pmatrix}$.

(i) (2 points) Find a basis for Row(A).

Row(A) = Col$\begin{pmatrix}-1 & 1 & 2\\ 2 & 1 & -1\end{pmatrix}$ which is easily seen to be $\mathbf{R}^2$. Thus, any set of two linearly independent vectors in $\mathbf{R}^2$ is a basis for Row(A), for example $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$.

(ii) (4 points) Find a basis for the orthogonal complement of Col(A).

We row-reduce:

$$\left(\begin{array}{ccc|c}-1 & 1 & 2 & 0\\ 2 & 1 & -1 & 0\end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 1 & 1 & 0\end{array}\right).$$

This gives $x_1 = x_3$, $x_2 = -x_3$, and x_3 free. One basis is $\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}$.

Problem 5.

The rest of the exam is free response. Unless otherwise instructed, show your work or you will receive little or no credit, even if your answer is correct.

a) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation of counterclockwise rotation by $90°$, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects across the line $y = x$.

(i) (2 points) Write the standard matrix A for T.

Enter your answer here: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(ii) (2 points) Write the standard matrix B for U. (no work necessary on this part)

Enter your answer here: $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(iii) (2 points) Find the standard matrix C for $U \circ T$.

Enter your answer here: $C = BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

b) (4 points, Unrelated to part (a)) "**Exam**" version

Suppose R and S are two linear transformations so that

$$R(v_1) = 2w_1, \qquad R(v_2) = -w_1 + w_2,$$

$$S(w_1) = z_1 + 2z_2, \qquad S(w_2) = 3z_2.$$

Find $(S \circ R)(v_1 - 2v_2)$. Fully simplify your answer in terms of the vectors z_1 and z_2.

Enter your answer here: $(S \circ R)(v_1 - 2v_2) = \boxed{4z_1 + 2z_2}$.

$$R(v_1 - 2v_2) = R(v_1) - 2R(v_2) = 2w_1 - 2(-w_1 + w_2) = 4w_1 - 2w_2$$

$$S(4w_1 - 2w_2) = 4S(w_1) - 2S(w_2) = (4z_1 + 8z_2) - 6z_2 = 4z_1 + 2z_2.$$

Problem 6.

Let $B = \begin{pmatrix} -7 & 8 \\ -4 & 5 \end{pmatrix}$.

a) (2 points) Find the eigenvalues of B.

Enter them here: $\lambda_1 =$ ________ $\lambda_2 =$ ________.

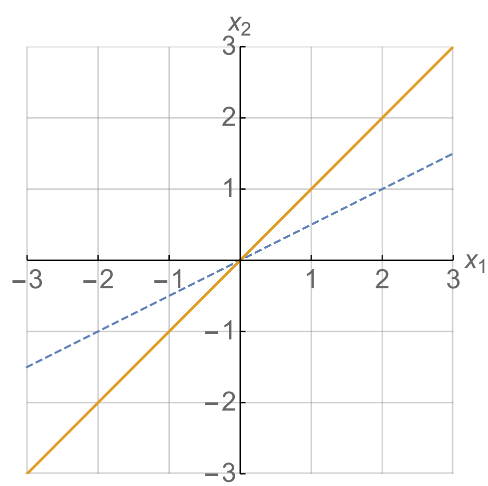
$$\det(B-\lambda I) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - (-2)\lambda + (-35+32)$$
$$= \lambda^2 + 2\lambda - 3 = (\lambda-1)(\lambda+3), \quad \text{so} \quad \lambda_1 = 1,\ \lambda_2 = -3.$$

b) (4 points) For each eigenvalue λ of B, find a corresponding eigenvector and draw the λ-eigenspace on the graph below. Clearly label each eigenspace.

$\left(B - I \mid 0 \right) = \left(\begin{array}{cc|c} -8 & 8 & 0 \\ -4 & 4 & 0 \end{array}\right) \longrightarrow RREF \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ gives 1-eigenspace Span $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, which is the line $x_1 = x_2$.

$\left(B + 3I \mid 0 \right) = \left(\begin{array}{cc|c} -4 & 8 & 0 \\ -4 & 8 & 0 \end{array}\right) \longrightarrow RREF \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ gives (-3)-eigenspace Span $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$, which is the line $x_1 = 2x_2$.

The solid line is the 1-eigenspace; the dotted line is the (-3)-eigenspace



c) (4 points) Find a formula for B^n, and simplify your answer completely.

$B = CDC^{-1}$ for $C = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$. $C^{-1} = \frac{1}{1-2}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$,

so

$$B^n = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-3)^n \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ (-3)^n & -(-3)^n \end{pmatrix} = \begin{pmatrix} -1+2(-3)^n & 2-2(-3)^n \\ -1+(-3)^n & 2-(-3)^n \end{pmatrix},$$

Problem 7.

Parts (a) and (b) are unrelated.

a) (6 points) Let $A = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$. Find all eigenvalues of A. For each eigenvalue, find one corresponding eigenvector. Enter your answers clearly in the spaces below.

$\lambda_1 = 2+i$; eigv. is $v_1 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$, $v_1 = \begin{pmatrix} -1 \\ -1+i \end{pmatrix}$, $v_1 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$, $v_1 = \begin{pmatrix} \frac{1+i}{2} \\ 1 \end{pmatrix}$.

$\lambda_2 = 2-i$; eigv. is $v_2 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ -1-i \end{pmatrix}$, $v_2 = \begin{pmatrix} 1-i \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} \frac{1-i}{2} \\ 1 \end{pmatrix}$.

Work is below.

$$\det(A - \lambda I) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 4\lambda + 5,$$

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i. \qquad \lambda_1 = 2+i, \quad \lambda_2 = 2-i.$$

<ins>For $\lambda = 2+i$</ins>: $(A - (2+i)I \ 0) = \left(\begin{array}{cc|c} 1-i & -1 & 0 \\ (*) & (*) & 0 \end{array}\right)$.

The 2×2 eigenvector trick for $\left(\begin{array}{cc|c} a & b & 0 \\ (*) & (*) & 0 \end{array}\right)$ gives $\begin{pmatrix} -b \\ a \end{pmatrix}$ as an eigenvector, so $v_1 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$. Or $v_1 = \begin{pmatrix} -1 \\ -1+i \end{pmatrix}$, or even $v_1 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$ or $v_1 = \begin{pmatrix} \frac{1+i}{2} \\ 1 \end{pmatrix}$.

<ins>For $v_2 = 2-i$</ins>: We would use the conjugate eigenvector fact that $v_2 = \overline{v_1}$, or we could solve directly: $(A - (2-i)I \ 0) = \left(\begin{array}{cc|c} 1+i & -1 & 0 \\ (*) & (*) & 0 \end{array}\right)$, so $v_2 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$. Or $v_2 = \begin{pmatrix} -1 \\ -1-i \end{pmatrix}$, or even $v_2 = \begin{pmatrix} 1-i \\ 2 \end{pmatrix}$ or $v_2 = \begin{pmatrix} \frac{1-i}{2} \\ 1 \end{pmatrix}$.

b) (4 points) **"Exam" version**

Let $A = \begin{pmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{pmatrix}$. As k goes to infinity, what vector does $A^k \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ approach? Enter your answer here: $A^k \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ approaches $\boxed{\begin{pmatrix} 3 \\ 5 \end{pmatrix}}$.

$A^k \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ approaches $(8+0)w$, where w is the steady-state vector for A. To get w, we find a 1-eigenvector and then scale:

$\left(A - I \mid 0\right) = \left(\begin{array}{c|c} -0.5 & 0.3 \\ 0.5 & -0.3 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -3/5 & 0 \\ 0 & 0 & 0 \end{array}\right)$. A 1-eigenvector is $v = \begin{pmatrix} 3/5 \\ 1 \end{pmatrix}$, so

$$w = \frac{1}{1+\frac{3}{5}} \begin{pmatrix} 3/5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/8 \\ 5/8 \end{pmatrix}, \qquad \text{so } A^k \text{ approaches } \quad 8w = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

Problem 8.

Consider the matrix A below and its RREF:

$$A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ -2 & -4 & -6 & 2 \\ 1 & 2 & -5 & -1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) (2 points) Write a basis for Col A. (no work necessary, no partial credit).

The pivot columns (1 and 3) form a basis for Col(A), but really column 3 and any other column will work.

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -6 \\ -5 \end{pmatrix} \right\}.$$

b) (4 points) Find a basis for Nul A. From the RREF of A, we see the solution set is

$$x_1 + 2x_2 - x_4 = 0, \quad x_3 = 0,$$

so $x_1 = -2x_2 + x_4$, x_2 and x_4 are free, and $x_3 = 0$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2 + x_4 \\ x_2 \\ 0 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \qquad \text{A basis is} \qquad \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

c) (1 point) Fill in the blank: $\dim\left((\text{Nul } A)^{\perp}\right) = 2$.

d) (3 points) Is there a matrix B so that $\text{Col}(B) = \text{Nul}(A)$? If yes, write such a B. If not, justify why no such matrix B exists.

Yes. Just take the columns of B to be a set whose span is Nul A, for example

$$B = \begin{pmatrix} -2 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 9.

Consider the subspace W of $\mathbf{R}^3$ that consists of all solutions $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ to the following system of equations:

$$\begin{aligned} x_1 - 3x_2 &= 0 \\ x_2 + x_3 &= 0. \end{aligned}$$

a) (4 points) Find the matrix B for orthogonal projection onto W.

A short computation shows $W = \mathrm{Span}\{u\}$ where $u = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$. Thus

$$B = \frac{1}{u \cdot u} u u^T = \frac{1}{9+1+1} \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} -3 & -1 & 1 \end{pmatrix} = \boxed{\frac{1}{11} \begin{pmatrix} 9 & 3 & -3 \\ 3 & 1 & -1 \\ -3 & -1 & 1 \end{pmatrix}}$$

b) (2 points) Let P be the matrix for orthogonal projection onto $W^\perp$. Write the eigenvalues of P. For each eigenvalue of P, write its geometric multiplicity. You do not need to show your work for this part.

$\lambda_1 = 1$ has geometric multiplicity 2 since $\dim(W^\perp) = 2$
$\lambda_2 = 0$ has geometric multiplicity 1 since $\dim(W) = 1$.

c) (4 points) **"Exam"** version

Let $x = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$. Find vectors x_W in W and $x_{W^\perp}$ in $W^\perp$ so that $x = x_W + x_{W^\perp}$. Enter your answers here: $\quad x_W = \begin{pmatrix} 12/11 \\ 4/11 \\ -4/11 \end{pmatrix}, \quad x_{W^\perp} = \begin{pmatrix} 10/11 \\ -4/11 \\ 26/11 \end{pmatrix}.$

$$x_W = Bx = \frac{1}{11} \begin{pmatrix} 9 & 3 & -3 \\ 3 & 1 & -1 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 12/11 \\ 4/11 \\ -4/11 \end{pmatrix}, \qquad x_{W^\perp} = x - x_W = \begin{pmatrix} 10/11 \\ -4/11 \\ 26/11 \end{pmatrix}$$

Problem 10.

In this problem, we follow the standard labeling of points in the xy-plane as (x, y).

Find the least-squares line $y = mx + b$ for the data points $(-2, 0)$, $(1, 18)$, and $(4, -6)$.

Enter your answer here: $y = \underline{\quad -1 \quad} x + \underline{\quad 5 \quad}$. You must show appropriate work. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

The (inconsistent) system of equations for the line through the three data points is:

$$\begin{aligned} 0 &= -2m + b \\ 18 &= m + b \\ -6 &= 4m + b. \end{aligned}$$

This is the matrix equation $Ax = b$ where $A = \begin{pmatrix} -2 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 18 \\ -6 \end{pmatrix}$.

We need to solve $A^T A \widehat{x} = A^T b$.

$$A^T A = \begin{pmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 21 & 3 \\ 3 & 3 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 18 \\ -6 \end{pmatrix} = \begin{pmatrix} -6 \\ 12 \end{pmatrix}.$$

We solve:

$$\left(A^T \;\; A \,\middle|\, A^T b \right) = \left(\begin{array}{cc|c} 21 & 3 & -6 \\ 3 & 3 & 12 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 3 & 3 & 12 \\ 21 & 3 & -6 \end{array}\right) \xrightarrow[\text{then } R_2 = R_2/3]{R_1 = R_1/3} \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 7 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 7R_1} \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -6 & -30 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1 - R_2]{R_2 = -\frac{R_2}{6}} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 5 \end{array}\right)$$

Thus $\widehat{x} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$. Since we arranged the entries to correspond to $\begin{pmatrix} m \\ b \end{pmatrix}$, this means

$$\boxed{y = -x + 5}.$$

Scrap paper. This page will not be graded under any circumstances.