Markov Chain, part 2

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1 The gambler's ruin problem

Consider the following problem.

Problem. Suppose that a gambler starts playing a game with an initial \$B bank roll. The game proceeds in turns, where at the end of each turn the gambler either wins \$1 with probability p, or loses \$1 with probability $q = 1 - p$. The player continues until he or she either makes it to \$N, or goes bankrupt with \$0. Determine the probability that the player eventually reaches the \$N.

We can represent this by a Markov chain having $N+1$ states representing the amount of money that the player has: either \$0, \$1, ..., or \$N.

The transition probabilities are given as follows: $P_{0,0} = 1$; $P_{N,N} = 1$; and $P_{i,i+1} = p$ and $P_{i,i-1} = q$ for $i = 1, 2, ..., N-1$.

The corresponding transition matrix is

$$P = \begin{bmatrix} 1 & q & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & q & 0 & 0 & \cdots & 0 & 0 \\ 0 & p & 0 & q & 0 & \cdots & 0 & 0 \\ 0 & 0 & p & 0 & q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & q & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & p & 1 \end{bmatrix}.$$

We will analyze this problem two different ways: first, we will derive the the probability of winning in an *ad hoc* way by playing around with the structure of the problem; and then we will derive the probability more systematically by analyzing the above transition matrix.

1.1 Solution 1

Let P_i denote the probability of winning (reaching \$N before going broke), given that we start with \$i. Note that $P_0 = 0$ and $P_N = 1$.

We observe that

$$P_i \;=\; pP_{i+1} + qP_{i-1}.$$

The term pP_{i+1} here accounts for the case where we win the first turn, and then at the start of the second turn we have $\$(i+1)$; and the term qP_{i-1} accounts for the case where we lose the first turn. We now rewrite this equation, using the fact that $p + q = 1$ as follows:

$$P_{i+1} - P_i \;=\; \frac{q}{p}(P_i - P_{i-1}).$$

And so, for $j = 1, 2, ..., N$ we have

$$\begin{aligned} P_j - P_{j-1} \;=\; \frac{q}{p}(P_{j-1} - P_{j-2}) \quad &= \quad \left(\frac{q}{p}\right)^2 (P_{j-2} - P_{j-3}) \;=\; \cdots \\ &= \quad \left(\frac{q}{p}\right)^{j-1} (P_1 - P_0) \\ &= \quad \left(\frac{q}{p}\right)^{j-1} P_1. \end{aligned}$$

So,

$$\begin{aligned} P_i \;=\; P_i - P_0 \quad &= \quad (P_i - P_{i-1}) + (P_{i-1} - P_{i-2}) + \cdots + (P_1 - P_0) \\ &= \quad P_1 \sum_{j=0}^{i-1} \left(\frac{q}{p}\right)^j \\ &= \quad \begin{cases} P_1 \frac{(q/p)^i - 1}{(q/p) - 1}, & \text{if } q \neq p; \\ iP_1, & \text{if } q = p = 1/2. \end{cases} \end{aligned}$$

Note that this holds for $i = 0, 1, 2, ..., N$.

To complete our calculations, we need to find P_1, and to do this we note that since $P_N = 1$, and also

$$P_N = \begin{cases} P_1 \frac{1-(q/p)^N}{1-(q/p)}, & \text{if } q \neq q; \\ P_1 N, & \text{if } q = p = 1/2. \end{cases}$$

we have that

$$P_1 \ = \ \begin{cases} \frac{1-(q/p)}{1-(q/p)^N}, & \text{if } q \neq p; \\ 1/N, & \text{if } q = p = 1/2. \end{cases}$$

So,

$$P_i \ = \ \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } q \neq p; \\ i/N, & \text{if } q = p = 1/2. \end{cases}$$

1.2 Solution 2

It turns out that we can also determine those probabilities P_i by working with the matrix P: the condition that starting with $\$i$ we eventually reach $\$N$ and stop is equivalent to a path through the transition diagram for the Markov chain starting at the node labeled i and ending at the node labeled N. Since once we reach node N we stay there (it has no edges to any other node), the probability we seek should be the entry in the $(i+1)$st column and $(N+1)$st row of the matrix

$$Q \ := \ \lim_{n \to \infty} P^n$$

Now, it's pretty clear that if we iteratively travel from node to node in the transition graph, we must eventually wind up in node 0 or node N, where we will remain stuck. So, we should have that

$$Q \ = \ \begin{bmatrix} 1 & 1-a_1 & 1-a_2 & 1-a_3 & 1-a_4 & \cdots & 1-a_{N-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_1 & a_2 & a_3 & a_4 & \cdots & a_{N-1} & 1 \end{bmatrix}.$$

Clearly, then, P has only $\lambda = 1$ as an eigenvalue of magnitude 1, and that eigenvalue occurs to multiplicity at least 2 since the first and last columns of P (or even Q) are eigenvectors. Note that the other columns of Q are linear combinations of those two principal eigenvectors.

It remains to solve for the $a_1, ..., a_{N-1}$: first, note that $PQ = Q$ will not give us anything; but strangely enough, $QP = Q$ *does* tell us something about the a_i's – it tells us that

$$1 - a_1 = q + (1 - a_3)p, \; q(1 - a_{N-2}) = 1 - a_{N-1},$$

and

$$a_i = qa_{i-1} + pa_{i+1}, \text{ for } i = 2, ..., N-2.$$

If you work out these equations, you find that

$$a_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } q \neq p; \\ i/N, & \text{if } q = p = 1/2. \end{cases}$$

Since a_i was the entry in the $(i+1)$st column, $(N+1)$st row of Q, which we agreed was the probability P_i, we are now done.