

Homework for Analytic Methods in Number Theory

January 16, 2007

1. Is the Prime Number Theorem (which asserts that $\pi(x) \sim x/\log x$) equivalent to the following estimate ? :

$$\sum_{\substack{p \leq x \\ p \text{ prime}}} \frac{1}{p} = \log \log x + C + o\left(\frac{1}{\log x}\right),$$

where C is a certain constant. If so, provide a proof.

2. In this problem I will walk you through the proof that

$$\tau(n) < 2^{(1+o(1)) \log n / \log \log n},$$

where $\tau(n)$ is the divisor function, which gives the number of positive integer divisors of the positive integer n .

Step A. Explain why any $n \leq x$ which maximizes $\tau(n)$ must be a product of consecutive primes to powers; that is, $n = 2^{a_2} 3^{a_3} \cdots p_k^{a_k}$, where $a_i \geq 1$ and p_k is the k th prime number.

Step B. Building on what was shown in step A, explain why $p_k \leq (1 + o(1)) \log x$.

Step C. Write $\tau(n) = \Delta_1 \Delta_2$, where

$$\Delta_1 = \prod_{\substack{p < (\log x) / (\log \log x)^3 \\ p \text{ prime}}} (a_p + 1), \quad \Delta_2 = \prod_{\substack{(\log x) / (\log \log x)^3 \leq p < p_k \\ p \text{ prime}}} (a_p + 1).$$

By finding bounds on a_p , explain why

$$\Delta_1 = 2^{o((\log x)/\log \log x)}.$$

Step D. Explain why

$$\Delta_2 \leq 2^{\sum_{\substack{(\log x)/(\log \log x)^3 \leq p \leq p_k \\ p \text{ prime}}} a_p}.$$

Step E. Show that

$$\sum_{\substack{(\log x)/(\log \log x)^3 \leq p \leq p_k \\ p \text{ prime}}} a_p < \frac{(1 + o(1)) \log x}{\log \log x}.$$

(Hint: What is an upper bound for this sum if the weight is $a_p \log p$, instead of a_p ?) Deduce the bound

$$\tau(n) < 2^{(1+o(1))(\log n)/\log \log n}.$$

3. Use whatever method you like to give a “good” estimate (perhaps with an error as good as $O(x \log x)$ or perhaps $O(x)$) for the sum

$$\sum_{n \leq x} \sigma(n),$$

where $\sigma(n)$ is the sum of divisors of n – i.e. $\sigma(n) = \sum_{d|n} d$. Two methods that come to mind are: The convolution method and the hyperbola method.