Concordance of Surfaces

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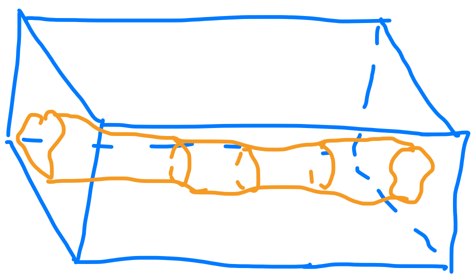
maybe joint w/ Michael Klug

Knots $S^1 \hookrightarrow M^3$
usually $K = J$ if they are isotopic



$M^3 \times I$

level preserving surface $\leftrightarrow$ isotopy

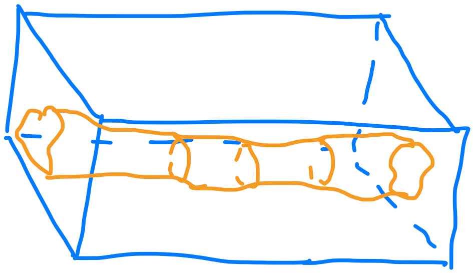
$K \sim J$ concordant if they cobound an annulus in $M \times I$

group $\mathcal{C} = \{[\text{knots in } S^3]\}/\text{concordance}$

$\longrightarrow\!\!\!\!\rightarrow \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$

Question: Does $\mathcal{C}$ have n torsion for $n \neq 2$?
Does $\mathcal{C}$ have ∞-divisible elements?

$S^2 \hookrightarrow M^4$
usually $S_0 = S_1$ if they are isotopic



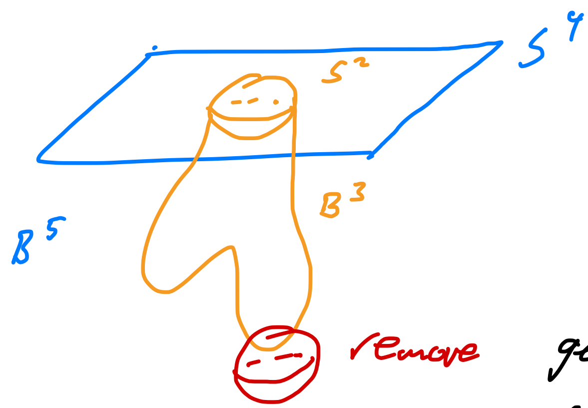
level preserving surface $\leftrightarrow$ isotopy

S_0, S_1 concordant if they cobound an $S^2 \times I$ in $M^4 \times I$

group $\mathcal{C}_4 = \{[S^2\text{'s in } S^4]\}/\text{concordance} = 1$

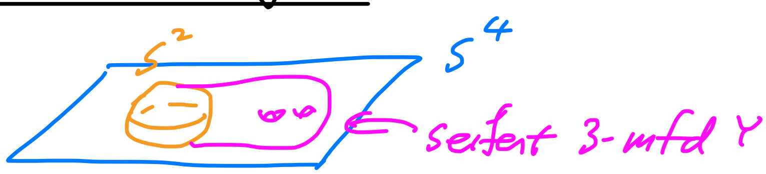
<u>Thm (Kervaire):</u>

every $S^2 \hookrightarrow S^4$ bounds a ball in B^5

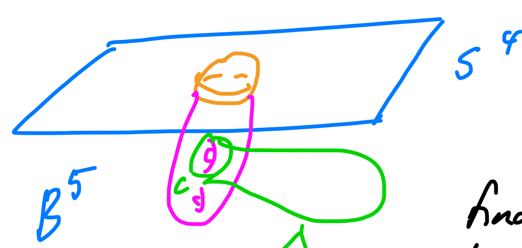


get $S^2 \times I$ in $S^4 \times I$
concordance S^2 to trivial S^2

<u>Pf (Kervaire-Sandjian):</u>



push Y into B^5



find c in Y
bounds Δ disk in B^5

do Dehn surgery along c

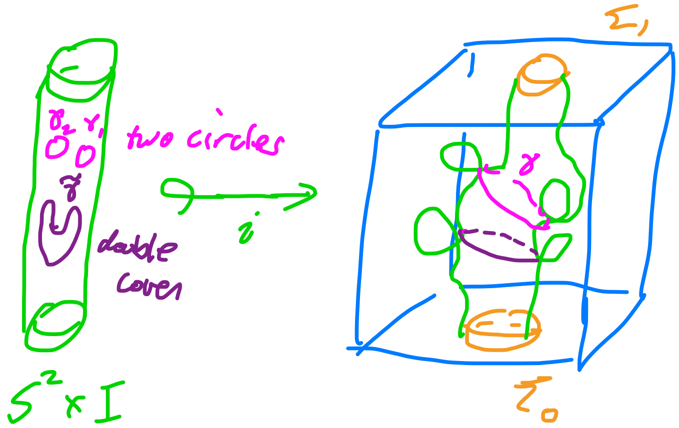
eventually becomes a ball ✓

Thm (Sandejian)

any two homologous surfaces genus-g surfaces Σ_0, Σ_1 in a $\pi_1 = 1$, X^4 are concordant

Goal: understand what property Σ_0, Σ_1 could have that would make them "look" not concordant

assume Σ_0, Σ_1 homotopic 2-spheres



$S^2 \times I$ immersed

loops of self-intersection

γ_1 bounds D in $S^2 \times I$

$i(D)$ is immersed disk so

γ is 1 in $\pi_1(X$

$\tilde{\gamma}$ bounds D in $S^2 \times I$

$i(\partial D) = g$ is order two in $\pi_1 X$

Def (FQ ST)

S_0, S_1 based homotopic S^2s in X

$fq(S_0, S_1) = \sum_{g \in H \cap H} g \in \mathbb{F}_2\{h^2 = 1, h \neq 1 \text{ in } \pi_1 X\}/\sim$ ← some equiv.

H a homotopy

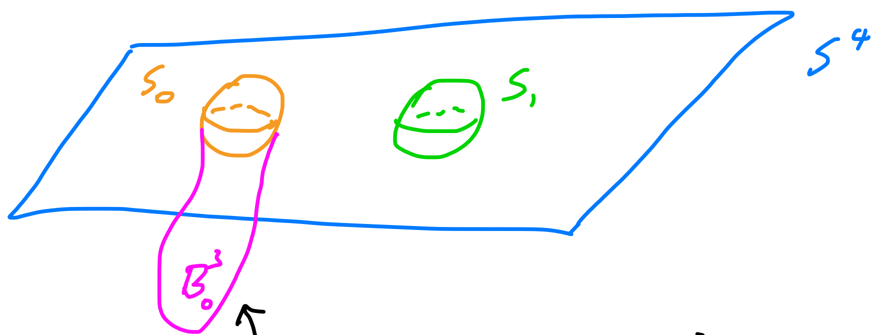
Example: Schwartz, FQ, ST

of 2-spheres in X^4 with π_1 having 2-torsion

homotopic but not concordant

Open Problem: is every link of S^2 in S^4 slice?

(i.e. bounds $\amalg_n B^3$ into B^5?)

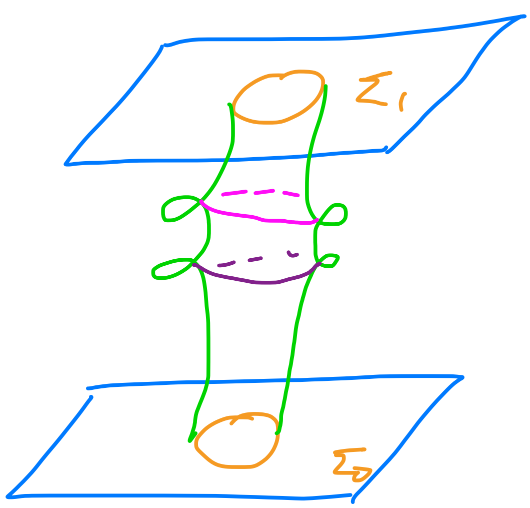


we know this exists

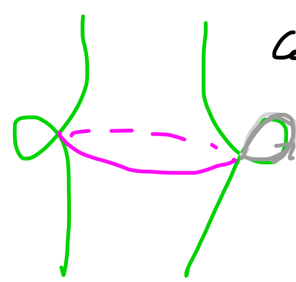
in $B^5 \setminus B_0^3$ does S_1 bound a ball?

$\uparrow$

homology $S^1 \times B^4$

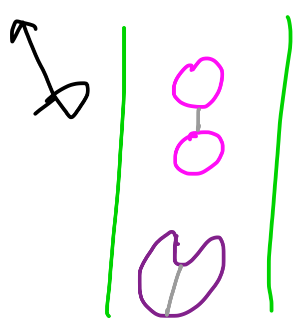


$X^4 \times I$



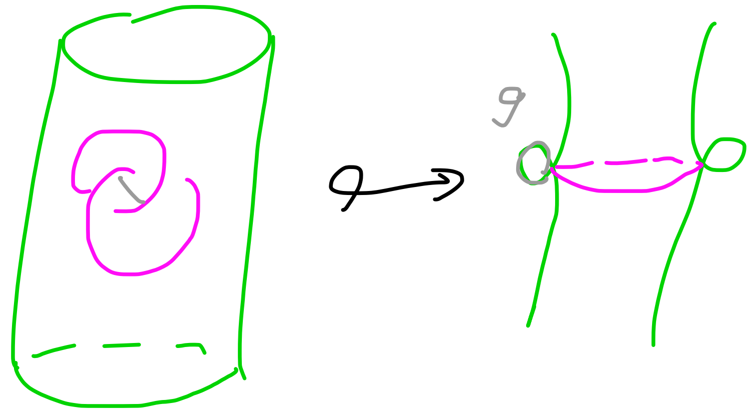
Consider

in preimage



or

specifically if preimage is



$[g] \in H_1(X; \mathbb{Z}/_2)$ is the Stong (S_0, S_1)

can you answer Open problem with this invt?

<u>Thm (KM):</u>

$\exists$ S_0, S_1 in 4-mfds whose π_1 has no 2-torsion

(e.g. $S^1 \times S^3 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$)

that are homotopic but not concordant

Thm (Gabai):

if $X^4 \supset R_0, R_1$ homotopic and $G \subset X$ a 2-sphere

s.t. $G \cap R_i = pt$

and $G \cdot G = 0$, $\pi_1 X$ no 2-torsion

then R_0, R_1 isotopic

$Th^{\underline{m}}$ (FQ, KM):

if $X^4 \supset R_0, R_1$ homotopic and $G \subset X$ a immersed 2-sphere

st. $G \cap R_i = pt$

and $G \cdot G = 0$ mod 2 ← necessary by $Th^{\underline{m}}$ (KM) above

$fq(R_0, R_1) = 0$ ← necessary by Schwartz, ST, KT

then R_0, R_1 concordant

Open Problem: How do we obstruct (or construct) concordance between Σ_0, Σ_1 if positive genus when

$$\pi_1 \Sigma_i \to \pi_1 X^4$$

is nontrivial?