The Thurston norm and a baby version: The Intersection norm

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Overview

- Thurston norms
- Foliations and Thurston norms
- Intersection norms
- Geography of Thurston balls and Intersection balls

Goal

Understand the relation between Thurston norms and Intersection norms

To start with

Theorem (Poincaré-Hopf)

If X is a nonsingular vector field on Σ_g, then $g = 0$.

Question

Which 3-manifolds admit "nice foliation"?

THURSTON NORMS

Definition:

M: Compact orientable 3-manifolds (with torus boundary components),

Fact:

Let $a \in H_2(M, \mathbb{Z})$. Then, a can be represented by embedded surfaces:

$$a = [\cup S_i].$$

Complexity: If $a = [\cup S_i]$, we set:

$$\chi_-(\cup S_i) = \sum_{i=1}^{m} \max\{0, -\chi(S_i)\}$$

and

$$T(a) = \inf_{[S]=a} \{\chi_-(S)\} \in \mathbb{N}.$$

We get a map:

$$\begin{aligned} T : H_2(M, \mathbb{Z}) \simeq \mathbb{Z}^k &\longrightarrow \mathbb{N} \\ a &\longmapsto \inf_{[S]=a} \{\chi_-(S)\} \end{aligned}$$

Theorem (Thurston)

- T extends to a semi-norm $T : H_2(M, \mathbb{R}) \simeq \mathbb{R}^k \to \mathbb{R}_+$
- If M is aspherical and atoroidal, then T is a norm.

Question

What is shape of the (dual) unit ball of T?

Theorem (W. Thurston)

The unit dual ball of T is an integer polytope:

$$B^1_{T^*} = \mathrm{ConvHull}\{w_1, ..., w_n\};$$

where $w_i \in H^2(M, \mathbb{Z}) \simeq \mathbb{Z}^k$ are integer cohomology classes.

Example (W. Thurston)

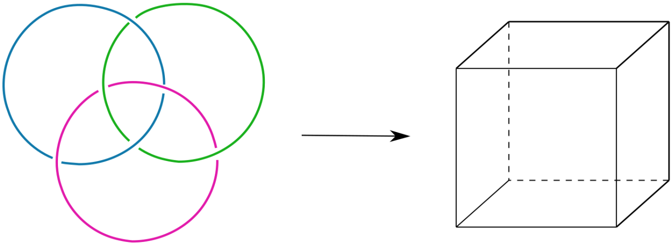


Figure: Borromean rings L and the unit dual ball of $\mathbb{S}^3 - L$

MOTIVATION: Foliations And Thurston Norms

Definition (Foliation)

A co-dimension 1 foliation $\mathcal{F}$ in M is a *locally trivial partition* of M by surfaces.

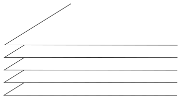


Figure: locally trivial foliation

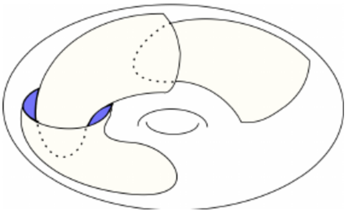


Figure: Reeb component: foliation on $\mathbb{D} \times \mathbb{S}^1$

Theorem (Lickorish-Novikov-Zieschang)

Every closed compact orientable 3-manifold admit a foliation (up to adding finitely many Reeb components).

Quote: (Foliations, A. Candel and L. Conlon) *"The principal moral to be drawn from this chapter is that since foliations with Reeb components are ubiquitous, they carry absolutely no information about the topology of 3-manifolds."*

Moral

Foliations without Reeb components are the most interesting for 3-manifolds.

Question

Which 3-manifolds admit transversally oriented foliation without Reeb components (nice foliation)?

The answer...

Theorems (W. Thurston)

- If $\mathcal{F}$ is a nice foliation on M then,

$$\text{Euler}(\mathcal{F}) \in B^1_{T^*}.$$

- If S is a compact leaf of $\mathcal{F}$, then S is minimizing; *i.e*

$$T([S]) = \chi_-(S).$$

Theorem (D. Gabai)

If S is minimizing and $[S] \neq 0$, then S is a leaf of a nice foliation on M.

Corollary (D. Gabai)

If $b_2(M) > 0$, then M admit a nice foliation.

Realization problem for Thurston norms

Question

Given an integer polytopes

$$P := \mathrm{ConvHull}\{v_1, ..., v_n\}$$

in $\mathbb{R}^k$*, is* P *a Thurston ball?*

INTERSECTION NORMS

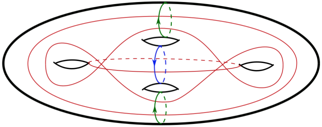
Definition

Σ_g: closed oriented surface of genus g,
$\gamma := \{c_1, ..., c_k\}$: collection of closed curves on Σ_g.

Fact:
If $a \in H_1(\Sigma_g, \mathbb{Z})$, then there is an oriented multicurve α such that $a = [\alpha]$.

We set,

$$i_\gamma(\alpha) := \text{card}\{\alpha \cap \gamma\}, \quad N_\gamma(a) := \inf_{[\alpha]=a} \{\#(\alpha \cap \gamma)\} \in \mathbb{N}.$$



So, we get a map

$$N_\gamma : H_1(\Sigma_g, \mathbb{Z}) \simeq \mathbb{Z}^{2g} \to \mathbb{N}$$

Theorem (M. Cossarini and P. Dehornoy)

- If γ is filling, N_γ extends to a norm $N_\gamma : H_1(\Sigma_g, \mathbb{R}) \to \mathbb{R}_+$.
- Unit dual ball of N_γ is an integer polytope

$$B^1_{N^*_\gamma} = \mathrm{ConvHull}\{\sigma_1, ..., \sigma_n\}$$

where $\sigma_i \in H^1(\Sigma_g, \mathbb{Z})$ satisfy the parity condition

$$\sigma_i = [\gamma] \mod 2.$$

Theorem (M. Cossarini and P. Dehornoy)

Unit dual balls of intersection norms classify open book decomposition of $T^1\Sigma_g$.

Realization problem for intersection norms

Question

Given an integer polytopes

$$P := \mathrm{ConvHull}\{v_1, ..., v_n\}$$

in $\mathbb{R}^{2g}$*, is* P *an intersection ball on* Σ_g*?*

GEOGRAPHY OF THURSTON BALLS AND INTERSECTION BALLS

Theorem (W. Thurston)

- Parity condition: If $P := \text{ConvHull}\{w_1, ..., w_n\}$ is a Thurston ball, then

$$w_i = w_j \mod 2.$$

- The parity condition is sufficient for symmetric integer polygon P in $\mathbb{R}^2$.

Theorem (S.)

- Every intersection ball is a Thurston ball.
- There are some integer polytopes in $\mathbb{R}^4$ satisfying the parity condition which are not intersection balls.

Question

Are there Thurston balls which are not intersection norms balls?

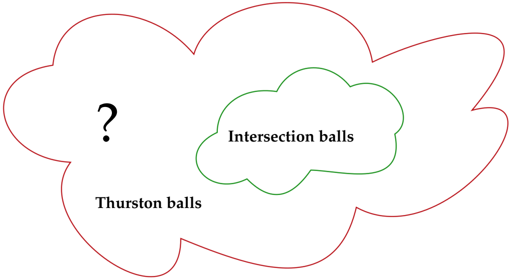


Figure: Geography of Thurston balls in even dimension

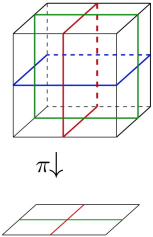
Sketch of Proofs

Thurston proof of the realization problem

Let P be an integer polygon that satisfies the parity condition.

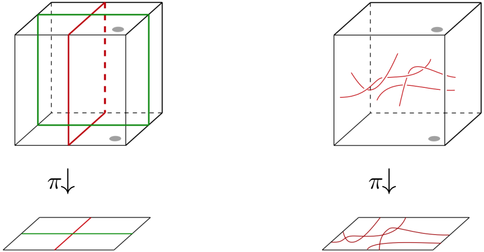
To construct M,...?

$\widetilde{M} := \mathbb{T}^3$; $\pi : \mathbb{T}^3 \to \mathbb{T}^2$ is a circle bundle.



Modification on $\widetilde{M}$

- Eliminate the horizontal surface with a Dehn surgery along a fiber,
- "Choose" an oriented closed curve γ on $\mathbb{T}^2$ such $[\gamma] \neq 0$
- Take a lift $\widetilde{\gamma}$ of γ in $\widetilde{M}$ and set $M = \widetilde{M} - \widetilde{\gamma}$



1. $\dim H_2(M) = 2$,
2. if α minimally intersects γ, $S_\alpha := \pi^{-1}(\alpha)$ is minimizing and
$$T([S]) = \operatorname{card}\{\alpha \cap \gamma.\}$$
3. $B^1_{T^*}$ is determined by the topology of γ.

Our proof

Theorem (S.)

Every intersection ball is a Thurston ball.

Key Idea

1. What Thurston showed is essentially that for every $a \in H_2(M, \mathbb{Z})$,

$$T(a) = N_\gamma(\pi_*(a)) \qquad (1);$$

$\pi_*(a) \in H_1(\mathbb{T}^2)$

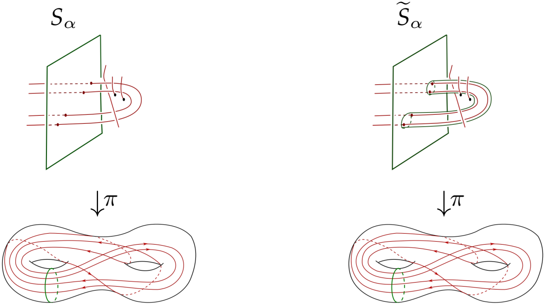
2. Equation (1) holds for $g \geq 2$ with slight modifications on Thurston construction.

OUR CONSTRUCTION

γ= filling closed curves on Σ_g.
M= the 3-manifold obtained following Thurston's construction.

Bad thing for $g \geq 2$

- *Attaching handle can reduce the complexity of vertical surfaces*



How to lift γ when $g \geq 2$?

How to avoid the situation where vertical surfaces are not minimizing?

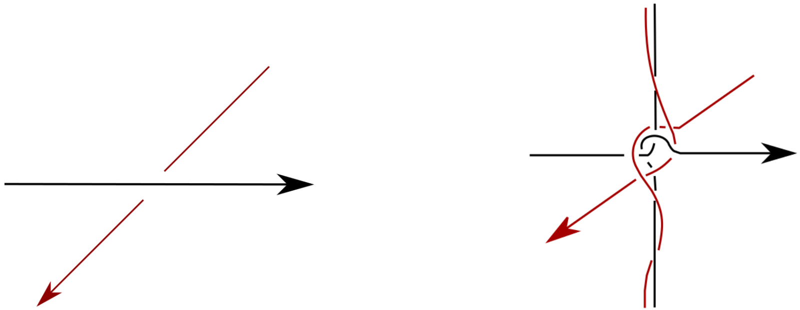


Figure: Braiding along fibers of double point of γ

... Vertical surface S_α are minimizing provided that α intersects γ minimally.

Question

- Are there Thurston balls which are not intersection balls?

- Let $v_1 := (1,1,1,1)$, $v_2 := (1,-1,1,1)$, $v_3 := (-1,1,1,1)$, $v_4 := (1,1,-1,1)$ and

$$P := \text{ConvHull}\{\pm v_i, i = 1, ..., 4\}.$$

  Is P a Thurston ball?
- What is the "topology" of 3-manifolds with smallest possible balls?

Thank you!